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Quasi-classical spectral series of the Dirac operators corresponding to quantized two-dimensional Lagrangian tori

V G Bagrov†, V V Belov‡, A Yu Trifonov† and A A Yevseyevich†

† High Current Electronics Institute, Russian Academy of Sciences, Siberian Division, Akademichesky Ave. 4, Tomsk, 634055, Russia

‡ Moscow Institute of Electronics and Mathematics, B. Vuzovsky 3/12, Moscow 109028, CIS, Russia

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Abstract. Based on the Maslov complex germ theory, a method of constructing the quasi-classical spectral series for the Dirac operator is proposed. The case when the corresponding relativistic Hamiltonian system is non-integrable and it admits a family of invariant two-dimensional stable Lagrangian tori containing the focal points is considered. The resulting quantization conditions for the above family generalize the Bohr–Sommerfeld–Maslov conditions and include new additional characteristics. The quasi-classical asymptotics obtained are regular over the full classically allowed domain. They also form an asymptotically complete and orthonormal set. Examples which use the proposed technique of the quasi-classical quantization are analysed.

1. Introduction

This paper is devoted to the problem of asymptotics (quasi-classical) quantization of non-integrable systems in the region of their regular motion [1]. In this case we failed to construct a family of invariant n-dimensional Lagrangian tori in a $2n$-dimensional phase space. (In mathematical literature the corresponding torus is termed isotropic.) Nevertheless, it often occurs that a non-integrable Lagrangian system possesses tori with a smaller dimension than that of the initial configuration space. Such a situation is typical of systems which possess a certain set of conserved quantities—the motion integrals [2]. Examples include a relativistic electron in the inhomogeneous magnetic field of an accelerator (with weak focusing), a hydrogen atom in a strong magnetic field (the Zeeman effect), etc.

A rigorous mathematical theory of the quasi-classical quantization of invariant incomplete-dimensional Lagrangian tori and constructing quasi-classical asymptotics (the so-called complex germ theory) was developed in general outline in [3, 4]. The basic idea of this theory is to reduce the initial problem of constructing asymptotic solutions to the study of geometric objects of the classical mechanics—the family of invariant Lagrangian tori with complex germ. From this family a discrete subfamily which generates a corresponding spectral series—a set of quasi-classical energy levels and corresponding to them quasi-classical eigenfunctions—is selected according to the quantization conditions (which together with the Maslov index contain new characteristics). Eigenfunctions form an asymptotically complete orthonormal set, and they are localized in a neighbourhood of a classically allowed region. In [5, 6] it was noted that the existence of a complex germ is equivalent to the orbit stability of the torus.

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In this paper the Maslov complex germ theory is applied to construct quasi-classical spectral series of the Dirac operator for the case when the relativistic classical system permits a family of invariant two-dimensional Lagrangian tori. (The Dirac operator spectral series corresponding to the motion of a relativistic electron along a closed stable orbit were obtained in [7].) Neglecting some particular technical details we shall qualitatively describe the main stages of the construction.

The following spectral problem is considered

$$(\hat{H}_{\mathrm{D}} - E)\Psi_E = 0 \tag{1.1}$$

where

$$\hat{H}_{\mathrm{D}} = H_{\mathrm{D}}\left(-\mathrm{i}\hbar\frac{\partial}{\partial q}, q, \hbar\right)$$

is the Weyl-ordered Dirac operator in an external electromagnetic field, E is the spectral parameter depending on $\hbar$, and $q = (q_1, q_2, q_3)$ are the coordinates (curvilinear in the general case) of the configuration space $\mathbb{R}^3_q$. Let the main symbol of the operator $\hat{H}_{\mathrm{D}}$ be denoted $\overset{\circ}{H}(p,q) = H_{\mathrm{D}}(p,q,0)$. The matrix $\overset{\circ}{H}(p,q)$ possesses two doubly degenerate eigenvalues $\lambda^{(\pm)}(p,q)$, one on which $\lambda^{(+)}(p,q)$ coincides with the classical Hamiltonian function of a relativistic electron. The corresponding classical motion is described by the Hamiltonian system

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\lambda^{(+)}_q \qquad \frac{\mathrm{d}q}{\mathrm{d}t} = \lambda^{(+)}_p. \tag{1.2}$$

The case of a partially integrable system (1.2) allowing a family of two-dimensional invariant Lagrangian tori is considered. This situation is typical when an electron moves in the fields with an axial symmetry.

Namely, let the variable $q_3 = \varphi(\mathrm{mod}\,2\pi)$ be cyclic. Then system (1.2) has the two integrals of motion: the integrals of energy $E_0 = \lambda^{(+)}(p,q)$ and momentum $I_0 = p_\varphi$. It is supposed that in a certain region of varying parameters $\omega = (E_0, I_0)$ in the phase space $\mathbb{R}^3_p \times \mathbb{R}^3_q$, system (1.2) permits a smooth two-parametric family of invariant two-dimensional Lagrangian tori $\Lambda^2(\omega) = \{(p,q) : p = p(\tau,\omega), q = q(\tau,\omega)\}$ lying on a joint surface of the energy level E_0 and the momentum I_0. The real coordinates $\tau = (\tau_1, \tau_2)$ on $\Lambda^2(\omega)$ are chosen in such a way that $\tau_1 = t$. It is also assumed that $\Lambda^2(\omega)$ is defined by the equations $p = p(\tau,\omega)$, $q = q(\tau,\omega)$ by means of only one set $\tau \in \mathbb{R}^2_\tau$.

According to the general theory of the complex germ [3, 4], to construct the asymptotics which correspond to the invariant Lagrangian manifold $\Lambda^2(\omega)$ it is not sufficient to know only the manifold $\Lambda^2(\omega)$. This makes up one of the principal distinctions from the case of the quasi-classical Maslov asymptotics with real phases [8, 9] to which the invariant Lagrangian tori of complete dimension (coinciding with that of configuration space) correspond. It is necessary to construct a new geometric object, a complex germ $r^3(\Lambda^2(\omega))$, which is responsible for the complex part of the asymptotic phase. In essence, $r^3(\Lambda^2(\omega))$ is given by a set of three linearly independent vector-functions $a_k(\tau)$, $k = 1, 2, 3$, being the solutions of the linear Hamiltonian system (which is derived from (1.2) by linearization in a neighbourhood of the manifold $\Lambda^2(\omega)$) and satisfying the conditions of Lagrangianity and dissipativity. As a result, we obtain a geometric object $[\Lambda^2(\omega), r^3(\Lambda^2(\omega))]$—a family of Lagrangian manifolds $\Lambda^2(\omega)$ with complex germ $r^3(\Lambda^2(\omega))$.

The following basic point in constructing the quasi-classical asymptotics of (1.1) corresponding to the family $\Lambda^2(\omega)$ is due to the presence on $\Lambda^2(\omega)$ of singular (focal) points with respect to projection onto the configuration space. It is well known that in the standard WKB method [10] the presence of focal points turns out to be an obstacle to construction of a unified regular asymptotic solution (which is valid over the full configuration space, including the focal points). Within the framework of complex germ theory this problem is solved by means of constructing a canonical operator with complex phase which, in fact, does determine the rule of matching the local asymptotics. The original version of this operator was proposed by Maslov [3]. Modification of the general construction of the canonical operator with complex phase in the case of incomplete-dimensional tori (manifolds) containing focal points was made in [5, 6].

As applied to our case, the procedure for constructing a regular asymptotic is as follows. The manifold $\Lambda^2(\omega)$ is covered by a set of neighbourhoods Ω_j forming a canonical atlas on it. Let $\pi_q(\Omega_j)$ be a part of the configuration space onto which the neighbourhood Ω_j is projected. For each domain $\pi_q(\Omega_j)$, a corresponding asymptotic $(\mathrm{mod}\, \mathrm{O}(\hbar^{3/2}))$ solution of (1.1) is found

$$(\hat{H}_\mathrm{D} - E)\Psi^j_E(q, \hbar) = \mathrm{O}(\hbar^{3/2}). \tag{1.3}$$

This approximation is sufficient to define the leading term $\overset{\circ}{\Psi}{}^j_E$ of the local asymptotic Ψ^j_E. On the basis of the functions $\overset{\circ}{\Psi}{}^j_E$, by the construction of the canonical operator $K_{\Lambda^2(\omega)}$, a multi-valued function $\overset{\circ}{\Psi}_E$ is built up on the configuration space. To avoid multi-valuedness, one should impose additional conditions leading to the quantization conditions of the family $[\Lambda^2(\omega), r^3(\Lambda^2(\omega))]$. Unlike the Lagrangian tori of complete dimension, which are quantized by the Bohr–Sommerfeld–Maslov rule [7], the above conditions contain additional characteristics due to the complex germ $r^3(\Lambda^2(\omega))$ (see [5, 6]).

As a result, from the continuous sequence of values of the spectral parameter E one selects a discrete set of energy levels $E_{N,l}(\hbar)$, where N, $l(\hbar)$ is a set of quantum numbers defined as

$$\lim_{\hbar\to 0} \hbar l(\hbar) = I_0 \qquad \lim_{\hbar\to 0} E_{N,l}(\hbar) = E_0. \tag{1.4}$$

Equations (1.4) point out the correspondence of this quasi-classical spectral series to the classical motion with the energy E_0 and the momentum I_0.

The asymptotic eigenfunctions $\overset{\circ}{\Psi}_{E_{N,l}}$ are localized in some neighbourhood of the projection of $\Lambda^2(\omega)$ onto $\mathbb{R}^3_q$ and with the accuracy to $\mathrm{O}(\hbar^{1/2})$ they form an asymptotically complete orthonormal set of states

$$(\overset{\circ}{\Psi}_{E_{N',l'}} \mid \overset{\circ}{\Psi}_{E_{N,l}})_\mathrm{D} = \int \mathrm{d}^3 q \sqrt{g}\, \overset{\circ}{\Psi}{}^+_{E',l'} \overset{\circ}{\Psi}_{E,l} = \delta_{NN'}\delta_{ll'} + \mathrm{O}(\hbar^{1/2}) \tag{1.5}$$

where $^+$ denotes Hermitian conjugation.

The sequence of numbers $E_{N,l}(\hbar)$ and that of the functions $\overset{\circ}{\Psi}_{E_{N,l}}$ constructed in this way are a quasi-classical spectral series of the Dirac operator $\hat{H}_\mathrm{D}$ to which, in the limit $\hbar \to 0$, the family of invariant Lagrangian tori $\Lambda^2(\omega)$ correspond.

To complete the picture, it is worth noting that the problem of quasi-classical approximation for the Dirac operator in the case when the corresponding relativistic Hamiltonian system permits a family of complete-dimensional Lagrangian tori, was solved in [9] by constructing the Maslov canonical operator with real phase. The same problem was dealt with in the work by Leray [11] but considered from the viewpoint of Lagrangian analysis. The method of 'gauge-invariant' construction of the quasi-classical eigenvalues and eigenfunctions for matrix differential operators proposed recently by Littlejohn and Flynn [12, 13] also deserves attention. Here, contrary to the traditional approach, the construction is carried out in the phase space with specially chosen gauge-invariant coordinates being, however, not canonical. Note that the quasi-classical quantization of closed stable orbit are also investigated in mathematical literature [14–17] (see also discussions in [20–22]).

The paper is organized in such a way as to make it self-contained as much as possible. In sections 2 and 3 all the necessary facts from the complex germ theory are given. In section 4 these results are illustrated by the example of a Hamiltonian system allowing the family of non-complete Lagrangian tori $\Lambda^2(\omega)$. In section 5 for $\Lambda^2(\omega)$ a corresponding spectral series of the Dirac operator is built up. The results obtained are applied to specific physical systems in section 6. Part of the necessary material is taken into the appendices.

2. Complex germ on the family of two-dimensional Lagrangian manifolds

Let Λ^2 be a compact two-dimensional Lagrangian manifold which is invariant with respect to Hamiltonian system (1.2) and lying on a surface energy level of classical Hamilton function $\lambda^+(p,q)$, $\lambda^+|_{\Lambda^2} = E_0$. Let I denote an increasing subsequence of the set $\{1,2,3\}$, and $\bar{I}$ an ordered subsequence of the same set supplementing I to $\{1,2,3\}$. The ordered set $y = (q_I, p_{\bar{I}})$ of the phase space coordinates (p,q) corresponds to the set $(I,\bar{I})$; e.g. if $I = \{2,3\}$ and $\bar{I} = \{1\}$ then $y = (p_1, q_2, q_3)$, and when $I = \{1,2,3\}$ and $\bar{I} = \{\phi\}$, we are likely to have $y = q = (q_1, q_2, q_3)$. Let Ω_I denote the domain on Λ^2 which is projected onto the coordinate plane $(q_I, p_{\bar{I}})$ smoothly and in a single-valued manner. In other words, in the region Ω_I the following condition is valid

$$\operatorname{rank}\left\|\frac{\partial y}{\partial \tau}\right\| \equiv \operatorname{rank}\frac{\partial(q_I, p_{\bar{I}})}{\partial(\tau_1, \tau_2)} = 2. \tag{2.1}$$

Since Λ^2 is a compact manifold, the following statement holds true: the manifold Λ^2 may be covered by a finite set of neighbourhoods Ω_{I_N} satisfying condition (2.1). Such a set of neighbourhoods $\{\Omega_{I_N}\}$ forms a canonical atlas on Λ^2 (see also [3]).

For a given ordered set of coordinates $y = (q_I, p_{\bar{I}})$, the ordered set of conjugate momenta is $p_y = (p_I, -q_{\bar{I}})$. Consequently, for the above examples we shall have $p_y = (-q_1, p_2, p_3)$ and $p_y = (p_1, p_2, p_3)$, respectively. The transformation from the coordinates (p,q) to (p_y, y) is canonical and determined by the symplectic (6×6)-matrix

$$G_I = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}. \tag{2.2}$$

Here, A and B are the diagonal (3×3)-matrices of the form $A_{ab} = \delta_{aI}\delta_{ab}$, $B_{ab} = \delta_{a\bar{I}}\delta_{ab}$, where

$$\delta_{aI(\bar{I})} = \begin{cases} 1 & a \in I(\bar{I}) \\ 0 & a \notin I(\bar{I}) \end{cases} \qquad a, b = 1, 2, 3.$$

The canonical transformations leave the Hamiltonian system invariant, so that for (1.2) we have

$$\dot{p}_y = -\tilde{\lambda}_y^{(+)} \qquad \dot{y} = \tilde{\lambda}_{p_y}^{(+)} \tag{2.3}$$

where $\tilde{\lambda}^{(+)}(p_y, y)$ is the function $\lambda^{(+)}(p, q)$ expressed in the (p_y, y) coordinates. (Here and below a dotted term implies a derivative with respect to τ_1.)

Let a six-component vector-function

$$a(\tau) = \begin{pmatrix} W(\tau) \\ Z(\tau) \end{pmatrix}$$

be a solution of the system in variations

$$\dot{a}(\tau) = \begin{pmatrix} -\lambda_{qp}^{(+)}(\tau) & -\lambda_{qq}^{(+)}(\tau) \\ \lambda_{pp}^{(+)}(\tau) & \lambda_{pq}^{(+)}(\tau) \end{pmatrix} a(\tau) \tag{2.4}$$

which is the result of the linearization of Hamiltonian system (1.2) in a neighbourhood of $\Lambda^2 = \{p = p(\tau), q = q(\tau), \tau \in \mathbb{R}_\tau^2\}$. (By the function $\lambda_{pq}^{(+)}(\tau)$ is meant the function $\lambda_{pq}^{(+)}(p, q)$ taken at $p = p(\tau)$ and $q = q(\tau)$.) The same treatment of canonical system (2.3) leads to the system in variations of the form

$$\dot{\tilde{a}}(\tau) = \begin{pmatrix} -\tilde{\lambda}_{yp_y}^{(+)}(\tau) & -\tilde{\lambda}_{yy}^{(+)}(\tau) \\ \tilde{\lambda}_{p_yp_y}^{(+)}(\tau) & \tilde{\lambda}_{p_yy}^{(+)}(\tau) \end{pmatrix} \tilde{a}(\tau). \tag{2.5}$$

Equations (2.4) and (2.5) are the linear Hamiltonian systems, and the symplectic transformation $\tilde{a}(\tau) = G_I a(\tau)$ is canonical for them. Whence it follows that to each solution of (2.4)

$$a(\tau) = \begin{pmatrix} W(\tau) \\ Z(\tau) \end{pmatrix}$$

there is a solution of (2.5)

$$\tilde{a}(\tau) = \begin{pmatrix} \tilde{W}(\tau) \\ \tilde{Z}(\tau) \end{pmatrix}$$

where

$$\begin{aligned} \tilde{W}(\tau) &= AZ(\tau) - BW(\tau) = \begin{pmatrix} W_I(\tau) \\ -Z_{\bar{I}}(\tau) \end{pmatrix} \\ \tilde{Z}(\tau) &= BZ(\tau) + AW(\tau) = \begin{pmatrix} Z_I(\tau) \\ W_{\bar{I}}(\tau) \end{pmatrix}. \end{aligned} \tag{2.6}$$

In addition, in view of the matrix G_I being symplectic, for any two solutions of (2.5) there is an identity

$$\{\tilde{a}, \tilde{b}\} = \{a, b\} \tag{2.7}$$

where $\{\ \}$ imply the anti-symmetric inner product.

Let us define the notion of a complex germ on the Lagrangian manifold Λ^2. The following condition will be assumed to be fulfilled: equation (2.4) permits a set of three smooth with respect to $\tau = (\tau_1, \tau_2) \in \mathbb{R}_\tau^2$, linearly independent solutions $a_k(\tau)$, $k = 1, 2, 3$ such that:

(*a*) The first two of them, $a_1(\tau)$, $a_2(\tau)$, define the basis for the plane tangent to the manifold Λ^2 at the point $\tau \in \Lambda^2$ and are of the form

$$a_1(\tau) = \begin{pmatrix} \dot{p}(\tau) \\ \dot{q}(\tau) \end{pmatrix} \qquad a_2(\tau) = \begin{pmatrix} p'(\tau) \\ q'(\tau) \end{pmatrix}. \tag{2.8}$$

(Here and below the derivative with respect to τ_2 is denoted by prime.)

(*b*) The solution

$$a_3(\tau) = \begin{pmatrix} W(\tau) \\ Z(\tau) \end{pmatrix}$$

is a complex and bounded, with respect to the variables $\tau = (\tau_1, \tau_2)$, vector function satisfying the condition

$$\{a_3(\tau), \overset{*}{a}_3(\tau)\} = 2\mathrm{i} \tag{2.9}$$

where $*$ denotes the complex conjugation.

(*c*) All three solutions $a_k(\tau)$ are skew-normal in pairs to each other

$$\{a_k(\tau), a_l(\tau)\} = 0. \tag{2.10}$$

At each point $\tau \in \Lambda^2$ the complex plane $r^3(\tau)$ spanned by the vectors $a_k(\tau)$ forms a complex germ. In view of (2.10) this plane is a Lagrangian plane, and vectors $a_k(\tau)$ form the basis for it. The family of planes $\{r^3(\tau), \tau \in \mathbb{R}^2_\tau\}$ define the complex germ $r^3(\Lambda^2)$ on the invariant Lagrangian manifold Λ^2.

Note 1. Vector $a_3(\tau)$ introduced above is not, in the general case, a single-valued function on Λ^2, since at different τ and τ' determining the same point on Λ^2, we generally have $a_3(\tau) \neq a_3(\tau')$. Nevertheless, assuming a certain inaccuracy, we call $\{r^3(\tau), \tau \in \mathbb{R}^2_\tau\}$ a complex germ on Λ^2.

It should be mentioned now that since G_l is a symplectic transformation, it transfers the Lagrangian plane into itself, then in defining the complex germ $r^3(\Lambda^2)$ instead of the solutions $a_k(\tau)$ of (2.4) satisfying conditions (2.8)–(2.10), we might take the linear-independent solutions $\tilde{a}_k(\tau) = G_l a_k(\tau)$ of (2.5) which form a new basis for the germ $r^3(\Lambda^2)$. The vectors $\tilde{a}_1(\tau)$ and $\tilde{a}_2(\tau)$ tangent to the manifold Λ^2 have, according to (2.6) the form

$$\tilde{a}_1(\tau) = \begin{pmatrix} \dot{p}_y \\ \dot{y} \end{pmatrix} \qquad \tilde{a}_2(\tau) = \begin{pmatrix} p'_y \\ y' \end{pmatrix} \tag{2.11}$$

and, in turn, from conditions (2.9), (2.10) and identity (2.7) it follows that

$$\{\tilde{a}_k(\tau), \tilde{a}_l(\tau)\} = 0 \qquad \{\tilde{a}_3(\tau), \overset{*}{\tilde{a}}_3(\tau)\} = 2\mathrm{i}. \tag{2.12}$$

Hereinafter by the complex germ in the neighbourhood Ω_l will be meant the set of vector $\tilde{a}_k(\tau)$ defined above.

The conditions stated in this section determine a geometric object $[\Lambda^2(\omega), r^3(\Lambda^2)]$—the Lagrangian manifold with complex germ—which plays a key role in constructing quasi-classical asymptotics with complex phase [4].

3. Functions on a family of Lagrangian manifolds with complex germ

Let $\Omega_I \subset \Lambda^2$ be some neighbourhood with the local coordinates $y = (q_I, p_{\bar{I}})$, $P_y = (p_I, -q_{\bar{I}})$ and the following objects be introduced. We shall build up square (3×3)-matrices from the components of the complex germ vectors $\tilde{a}_k(\tau)$ in the neighbourhood Ω_I

$$\tilde{B}(\tau) = (\dot{p}_y(\tau), p'_y(\tau), \tilde{W}(\tau)) \qquad \tilde{C}(\tau) = (\dot{y}(\tau), y'(\tau), \tilde{Z}(\tau)). \tag{3.1}$$

Then, as follows from condition (2.1), in the region Ω_I the matrix $\tilde{C}(\tau)$ is non-singular. In this way, one can define the symmetric matrix $\tilde{Q} = \tilde{B}\tilde{C}^{-1}$ with the positively semi-defined imaginary part [6]: $\operatorname{Im} \tilde{Q}(\tau) \geqslant 0$.

Introduce a set of functions $\tau(y) = \{\tau_1(y), \tau_2(y)\}$ obeying the equations

$$\langle \dot{y}(\tau), \partial_y \tau_1 \rangle \bigg|_{\tau=\tau(y)} = 1 \qquad \langle \dot{y}(\tau), \partial_y \tau_2 \rangle \bigg|_{\tau=\tau(y)} = 0. \tag{3.2}$$

(Here and below the brackets $\langle \cdot, \cdot \rangle$ imply a scalar product in $\mathbb{R}^3$.) The existence of such a set, at least locally, follows from condition (2.1).

By Δy and $\Delta \hat{p}_y$ denote the operators

$$\Delta y = y - y(\tau) \qquad \Delta \hat{p}_y = -\mathrm{i}\hbar \partial_y \bigg|_{\tau=\text{const}} - p_y(\tau). \tag{3.3}$$

In the neighbourhood Ω_I introduce a complex action (complex phase)

$$\tilde{S}(y, \tau(y)) = \bigg[(E - E_0)\tau_1 + \int_{\tau_0}^{\tau} \langle p(\tau), \mathrm{d}q(\tau) \rangle - \langle q_{\bar{I}}(\tau), p_{\bar{I}}(\tau) \rangle + \langle p_y(\tau), \Delta y \rangle$$
$$+ \tfrac{1}{2} \langle \Delta y, \tilde{Q}(\tau) \Delta y \rangle \bigg]_{\tau=\tau(y)} \tag{3.4}$$

where $E = E_0 + \hbar E_1 + O(\hbar^2)$, $\lambda^{(+)}|_{\Lambda^2} = E_0$ and the integration in (3.4) is carried out along an arbitrary path on Λ^2 with the end at the point $\tau \in \Omega_I$.

Relate operators to the vectors $\tilde{a}_k(\tau)$ of the complex germ as follows

$$\begin{aligned} \hat{\tilde{a}}_1 &= \langle \dot{y}(\tau), \Delta \hat{p}_y \rangle - \langle \dot{p}_y(\tau), \Delta y \rangle & \hat{\tilde{a}}_3 &= \frac{1}{\sqrt{2\hbar}} (\langle \tilde{Z}(\tau), \Delta \hat{p}_y \rangle - \langle \tilde{W}(\tau), \Delta y \rangle) \\ \hat{\tilde{a}}_2 &= \langle y'(\tau), \Delta \hat{p}_y \rangle - \langle p'_y(\tau), \Delta y \rangle & \hat{\tilde{a}}_3^+ &= \frac{1}{\sqrt{2\hbar}} (\langle \overset{*}{\tilde{Z}}(\tau), \Delta \hat{p}_y \rangle - \langle \overset{*}{\tilde{W}}(\tau), \Delta y \rangle). \end{aligned} \tag{3.5}$$

In view of (2.12), the following commutation relations hold

$$[\hat{\tilde{a}}_k(\tau), \hat{\tilde{a}}_l(\tau)] = 0 \qquad [\hat{\tilde{a}}_3(\tau), \hat{\tilde{a}}_3^+(\tau)] = 1 \qquad k, l = 1, 2, 3. \tag{3.6}$$

Define the function $\tilde{J}(\tau) = \det \tilde{C}(\tau)$. Since the condition $\tilde{J}(\tau) \neq 0$ is fulfilled in the neighbourhood Ω_I, one can introduce the function

$$|0, \tau(y)\rangle = \frac{1}{\sqrt{\tilde{J}(\tau(y))}} \exp\left[\frac{\mathrm{i}}{\hbar} \tilde{S}(y, \tau(y))\right]. \tag{3.7}$$

Using the creation operator $\hat{\hat{a}}_3^+$ one can construct a set of functions of the form

$$|\nu, \tau(y)\rangle = \frac{1}{\sqrt{\nu!}}(\hat{\hat{a}}_3^+)^\nu |0, \tau(y)\rangle. \tag{3.8}$$

Using (3.4) and (3.6) it is not difficult to show that

$$\hat{\hat{a}}_1 |\nu, \tau(y)\rangle = 0 \qquad \hat{\hat{a}}_2 |\nu, \tau(y)\rangle = 0. \tag{3.9}$$

Finally, associate the Hamiltonian function $\tilde{\lambda}^{(+)}(p_y, y)$ to the Weyl-ordered quadratic operator in a mixed y-representation

$$\begin{aligned}\hat{\hat{\lambda}}_0^{(+)} = E_0 + \hat{\hat{a}}_1 + \tfrac{1}{2}\Big\{&\langle \Delta y, \tilde{\lambda}_{qq}^{(+)}(\tau)\Delta y\rangle + \langle \Delta y, \tilde{\lambda}_{yp_y}^{(+)}(\tau)\Delta \hat{p}_y\rangle + \langle \Delta \hat{p}_y, \tilde{\lambda}_{p_y y}^{(+)}(\tau)\Delta y\rangle \\ &+ \langle \Delta \hat{p}_y, \tilde{\lambda}_{p_y p_y}^{(+)}(\tau)\Delta p_y\rangle\Big\}_{\tau=\tau(y)}.\end{aligned} \tag{3.10}$$

Then, the functions (3.8) form a set of exact solutions of the Schrödinger equation with Hamiltonian (3.10)

$$(-i\hbar\partial_{\tau_1} + \hat{\hat{\lambda}}_0^{(+)} - E)|\nu, \tau(y)\rangle = 0. \tag{3.11}$$

As one can see below the functions $|\nu, \tau(y)\rangle$ are the basis for construction of the Maslov canonical operator with complex phase.

4. Family of invariant two-dimensional Lagrangian tori (special case)

The results of the two previous sections will be illustrated by an example of a partially integrable Hamiltonian system with an axial symmetry allowing a family of invariant two-dimensional Lagrangian tori $\Lambda^2(\omega)$, $\omega = (E_0, I_0)$.

Let the Hamiltonian function on the phase space $\mathbb{R}_p^3 \times \mathbb{R}_q^3$ have the form

$$\lambda^{(+)}(p, q) = \lambda^{(+)}(p_1, p_2^2, p_\varphi, q_1, q_2^2). \tag{4.1}$$

In view of the variable $q_3 = \varphi (\mathrm{mod}\, 2\pi)$ being cyclic, Hamiltonian system (1.2) permits the two integrals of motion

$$p_\varphi = I_0 \qquad \lambda^{(+)}(p_1, p_2^2, I_0, q_1, q_2^2) = E_0. \tag{4.2}$$

For given values of the energy E_0 and the momentum I_0 the set of equations (4.2) govern some joint surface $M(E_0, I_0)$. Let the surface $M(E_0, I_0)$ be connected and compact in *some region of the varying parameters E_0 and I_0. We shall consider its intersection with* the coordinate plane $\mathcal{T} = \{(p, q) : p_2 = q_2 = 0\}$. Then, as follows from (1.2), the intersection $M(E_0, I_0) \cap \mathcal{T}$ is the invariant Lagrangian manifold $\Lambda^2(E_0, I_0)$ of the form

$$\Lambda^2(E_0, I_0) = S^1(I_0) \times \Lambda^1(E_0, I_0) \tag{4.3}$$

where

$$\Lambda^1(E_0, I_0) = \{(p, q) : \lambda^{(+)}(p_1, 0, I_0, q_1, 0) = E_0\} \tag{4.4}$$

$$S^1(I_0) = \{(p_\varphi, \varphi) : p_\varphi = I_0, \varphi \in [0, 2\pi]\}. \tag{4.5}$$

Owing to the assumption of $M(E_0, I_0)$ being compact the curve $\Lambda^1(E_0, I_0)$ is closed, whence it follows that $\Lambda^2(E_0, I_0)$ forms a family of invariant two-dimensional Lagrangian tori.

The closed curve $\Lambda^1(E_0, I_0)$ obeys the pair of canonical equations

$$\dot{p}_1 = -\overset{\circ}{\lambda}{}^{(+)}_{q_1}(p_1, q_1, I_0) \qquad \dot{q}_1 = \overset{\circ}{\lambda}{}^{(+)}_{p_1}(p_1, q_1, I_0) \tag{4.6}$$

where $\overset{\circ}{\lambda}{}^{(+)}(p_1, q_1, I_0) = \lambda^{(+)}(p_1, 0, I_0, q_1, 0)$. In the phase plane (p_1, q_1), equations (4.6) describe a one-dimensional finite motion on the energy level E_0. By $\{p_1(\tau_1, \omega), q_1(\tau_1, \omega), \omega(E_0, I_0)\}$ denote a periodic (with respect to τ_1) solution of (4.6) with period T_1 given by

$$T_1 = 2\int_{R_-(\omega)}^{R_+(\omega)} \frac{\mathrm{d}q_1}{\overset{\circ}{\lambda}{}^{(+)}_{p_1}(q_1, E_0, I_0)} \tag{4.7}$$

where $R_\pm(\omega)$ are the roots of the equation $\overset{\circ}{\lambda}{}^{(+)}_{p_1}(q_1, E_0, I_0) = 0$, $\overset{\circ}{\lambda}{}^{(+)}_{p_1}(q_1, E_0, I_0) = \overset{\circ}{\lambda}{}^{(+)}_{p_1}(q_1, p_1(q_1, E_1, I_0), I_0)$, $p_1(q_1, E_0, I_0)$ we obtain from the equation $\overset{\circ}{\lambda}{}^{(+)}(p_1, q_1, I_0) = E_0$.

As far as the angular variable φ is concerned, there is the following formula for it with allowance for T_1-periodicity of the functions $p_1(\tau_1, \omega)$ and $q_1(\tau_1, \omega)$:

$$\varphi(\tau, \omega) = \int_{\overset{\circ}{\tau}_1}^{\tau_1} \overset{\circ}{\lambda}{}^{(+)}_{p_\varphi}(t, E_0, I_0)\, \mathrm{d}t + \tau_2 = \beta_1\tau_1 + \theta(\tau_1, \omega) + \tau_2 \tag{4.8}$$

where $\tau_2 (\bmod 2\pi)$,

$$\beta_1 = \frac{1}{T_1}\int_0^{T_1} \overset{\circ}{\lambda}{}^{(+)}_{p_\varphi}(t, E_0, I_0)\, \mathrm{d}t$$

and $\theta(\tau_1, \omega)$ is the periodic function with period T_1. The point $\overset{\circ}{\tau}_1$ marks the 'time origin' for Hamiltonian system (4.6).

Now we shall find the three vectors $a_k(\tau)$ forming the complex germ on the family $\Lambda^2(\omega)$ (4.3)–(4.5). The first two of them make up a symplectic basis for the tangent space of the manifold $\Lambda^2(\omega)$ and, according to (2.8), are as follows (for the sake of simplicity we shall omit the dependence on the parameters E_0 and I_0, wherever it does not cause confusion):

$$\begin{aligned} a_1(\tau_1) &= (\dot{p}_1(\tau_1), 0, 0, \dot{q}_1(\tau_1), 0, \dot{\varphi}(\tau_1))^T \\ a_2(\tau_1) &= (0, 0, 0, 0, 0, 1)^T. \end{aligned} \tag{4.9}$$

The third vector $a_3(\tau)$, which is skew-normal to the first two, will be described as

$$a_3(\tau_1) = (0, w(\tau_1), 0, 0, z(\tau_1), 0)^T. \tag{4.10}$$

Then, substituting (4.10) into (2.4) we obtain the following system of equations for a pair of complex functions $w(\tau_1)$ and $z(\tau_1)$

$$\begin{pmatrix} \dot{w} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & -\overset{\circ}{\lambda}{}^{(+)}_{q_2q_2}(\tau_1) \\ \overset{\circ}{\lambda}{}^{(+)}_{p_2p_2}(\tau_1) & 0 \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix}. \tag{4.11}$$

Additionally, from (2.9) it follows that

$$\mathrm{Im}(w\overset{*}{z}) = 1. \tag{4.12}$$

Equation (4.11) is the linear Hamiltonian system with T_1-periodic, with respect to τ_1, coefficients.

The Floquet solution is chosen as a solution of this equation, i.e.

$$\begin{pmatrix} w(\tau_1 + T_1) \\ z(\tau_1 + T_1) \end{pmatrix} = \mathrm{e}^{\mathrm{i}\Omega_1 T_1} \begin{pmatrix} w(\tau_1) \\ z(\tau_1) \end{pmatrix}. \tag{4.13}$$

In this case, the condition of reality of the Floquet characteristic index Ω_1 follows from (4.12). In its turn, the condition $\mathrm{Im}\,\Omega_1 = 0$ ensures that the vector-function (4.10) is limited in the entire region of varying the parameter $\tau_1 \in \mathbb{R}^1$. Thus, the vectors (4.9) together with the vector (4.10) satisfying (4.12) and (4.13) form the complex germ $r^3(\Lambda^2(\omega))$.

It should be noted that the number Ω_1, according to (4.13) is, generally speaking, determined with an accuracy to

$$\frac{2\pi}{T_1} k, k \in \mathbb{Z}.$$

Subsequently, we shall assume that the number Ω_1 was chosen in such a way that

$$\arg z(\tau_1 + T_1) = \arg z(\tau_1) + \Omega_1 T_1. \tag{4.14}$$

Now we shall proceed to constructing the canonical atlas on the manifold (4.3) and a set of functions (3.8) corresponding to it. It should be reminded that a point $r(\tau) \in \Lambda^2$ is called non-singular, if rank $\|\partial q/\partial\tau\| = 2$ at the point. On the other hand, a point $r(\tau)$ is called singular (focal). Similarly, if a neighbourhood $\Omega \subset \Lambda^2$ consists of non-singular points, then such a neighbourhood is called non-singular. Otherwise, we have a singular (focal) one.

In the case of the family $\Lambda^2(\omega)$ of the type (4.3)–(4.5) the set of all the singular points $\Sigma \subset \Lambda^2(\omega)$ at which the rank of the matrix

$$\left\| \frac{\partial q}{\partial \tau} \right\| = \begin{pmatrix} \dot{q}_1 & 0 & \dot{\varphi} \\ 0 & 0 & 1 \end{pmatrix}$$

is less than two consists of the points $\Sigma = \{r(\tau) \in \Lambda^2(\omega), \tau = (\tau_1^-, \tau_2) \cup (\tau_1^+, \tau_2), \tau_2 \in [0, 2\pi]\}$ where $\tau_1^\pm$ are zeros of the function $\dot{q}_1(\tau_1)$, i.e. $q_1(\tau_1^\pm) = R_\pm$. The projection $\pi_q(\Sigma)$ of these points onto the configuration space $\mathbb{R}^3_q$ forms the caustic $\pi_q(\Sigma) = S^1_+ \cup S^1_-$ made up of the two circles $S^1_\pm = \{(q), q_1 = R_\pm, q_2 = 0, \varphi \in [0, 2\pi]\}$. The closed curve $\Lambda^1(E_0, I_0)$ lying on the coordinate plane (p_1, q_1) is oriented counterclockwise and, for definiteness sake, assume that $\mathring{\tau}_1 < \tau_1^+ < \tau_1^- < \mathring{\tau}_1 + T_1$, $R_- < R_+$. Cover the curve $\Lambda^1(E_0, I_0)$ with the four neighbourhoods V_j, $j = \overline{1, 4}$ as shown in figure 1. Then the canonical atlas on $\Lambda^2(\omega)$ may be made up from four neighbourhoods of the type $\Omega_j = V_j \times \{\tau_2\}$, $\tau_2 \in [0, 2\pi]$. With such a choice, the neighbourhoods Ω_1, Ω_3 are non-singular, and the neighbourhoods Ω_2, Ω_4 are singular. The ordered set of the phase space coordinates $y = (q_I, p_{\bar{I}})$, where $I = \{2.3\}$, correspond to the neighbourhoods Ω_j, $j = 2, 4$, since in this case

$$\mathrm{rank} \left\| \frac{\partial q}{\partial \tau} \right\| = \mathrm{rank} \begin{pmatrix} \dot{q}_1 & 0 & \dot{\varphi} \\ 0 & 0 & 1 \end{pmatrix} = 2.$$

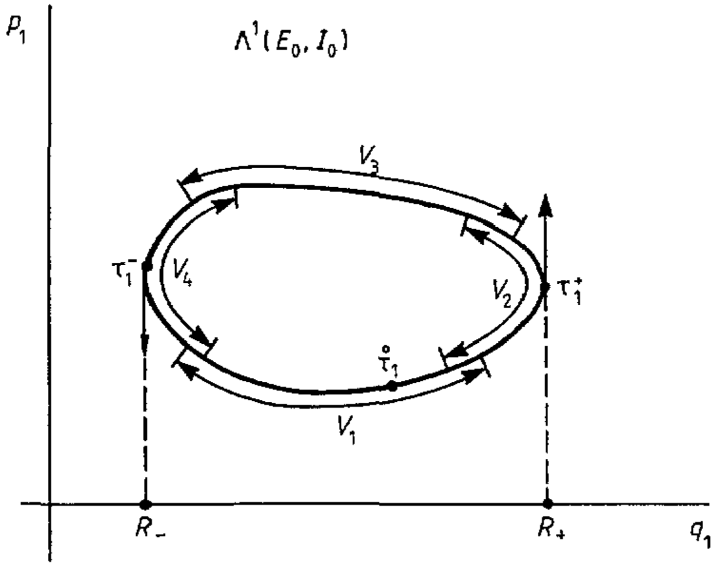

Figure 1. The canonical atlas on $\Lambda^1(E_0, I_0)$.

Now we shall give the explicit expressions for functions (3.8). Having omitted simple but rather cumbersome calculations we write out the final results. For the functions $|\nu, \tau^j(y)\rangle$ corresponding to the non-singular neighbourhoods Ω_j, $j = 1, 3$ we obtain

$$|\nu, \tau_1^j(q_1)\rangle = \left\{\left(\frac{1}{|\dot{q}_1(\tau_1)|^{1/2}} \exp\frac{\mathrm{i}}{\hbar}\int_{\overset{\circ}{\tau}_1}^{\tau_1} p_1(t)\,\mathrm{d}q_1(t)\right.\right.$$
$$\left.\left.\times \exp\frac{\mathrm{i}}{\hbar}\left[I_0\varphi + (E - E_0)\tau_1 + \tfrac{1}{2}\operatorname{Re}\frac{w(\tau_1)}{z(\tau_1)}q_2^2\right]\right)\Xi_\nu(q_2, \tau_1)\right\}\Bigg|_{\tau_1=\tau_1^j(q_1)} \tag{4.15}$$

where the functions $\Xi_\nu(q_2, \tau_1)$ are of the form

$$\Xi_\nu(q_2, \tau_1) = \frac{1}{\sqrt{\nu!}}\left(\frac{\mathrm{i}}{\sqrt{2}}\right)^\nu \frac{\mathrm{e}^{-\mathrm{i}\nu(\arg z(\tau_1))}}{\sqrt{-z(\tau_1)}}\mathrm{e}^{-q_2^2/[2\hbar|z(\tau_1)|^2]}H_\nu\left(\frac{q_2}{\sqrt{\hbar}|z(\tau_1)|}\right). \tag{4.16}$$

Here $H_\nu(\xi)$ are the Hermite polynomials. The integral in (4.15) is taken along a path on $\Lambda^1(E_0, I_0)$ oriented counterclockwise with the origin in the point $\overset{\circ}{\tau}_1 \in V_1$ and the end in the point $\tau_1 \in V_j$, $j = 1, 3$. The function $\tau_1^j(q_1)$ is the solution of the equation $q_1(\tau_1^j(q_1)) = q_1$ in the non-singular neighbourhood V_j. To construct states (4.15) the functions

$$\tau_2^j(q) = \varphi - \int_{\overset{\circ}{\tau}_1}^{\tau_1^j(q_1)} \overset{\circ}{\lambda}{}^{(+)}_{p_\varphi}(t)\,\mathrm{d}t$$

were used.

In the case of the focal neighbourhood Ω_j, $j = 2, 4$, the corresponding functions $|\nu, \tau^j(y)\rangle$ are

$$|\nu, \tau_1^j(p_1)\rangle = \left\{\left(\frac{1}{|\dot{p}_1(\tau_1)|^{1/2}} \exp\frac{\mathrm{i}}{\hbar}\left[\int_{\overset{\circ}{\tau}_1}^{\tau_1} p_1(t)\,\mathrm{d}q_1(t) - p_1q_1(\tau_1)\right]\right.\right.$$
$$\left.\left.\times \exp\frac{\mathrm{i}}{\hbar}\left[I_0\varphi + (E - E_0)\tau_1 + \frac{1}{2}\operatorname{Re}\frac{w(\tau_1)}{z(\tau_1)}q_2^2\right]\right)\Xi_\nu(q_2, \tau_1)\right\}\Bigg|_{\tau_1=\tau_1^j(p_1)}. \tag{4.17}$$

Here, as in the previous case, the integration is carried out along the oriented path on $\Lambda^1(E_0, I_0)$ with the origin and the end in the points $\overset{\circ}{\tau}_1 \in V_1$ and $\tau_1 \in V_j$, $j = 2, 4$, respectively. The functions $\tau_1^j(p_1)$ satisfy the equations $p_1(\tau_1^j(p_1)) = p_1$ and as the version of the functions $\tau_2^j(y)$, the functions

$$\tau_2^j(y) = \varphi - \int_{\overset{\circ}{\tau}_1}^{\tau_1^j(p_1)} \overset{\circ}{\lambda}{}^{(+)}_{p_\varphi}(t)\,\mathrm{d}t$$

were taken.

Note. To derive formulae (4.15) and (4.17) we have used the identity $\varphi(\tau^j(y)) = \varphi$, whose validity follows from the definition of the functions $\tau^j(y)$ and (4.8) for all Ω_j.

5. Quasi-classical spectral series of the Dirac operator corresponding to the family of invariant Lagrangian tori $\Lambda^2(\omega)$ with complex germ $r^3(\Lambda^2(\omega))$

The results obtained in the previous section enable us to proceed to constructing the quasi-classical spectral series $[\overset{\circ}{\Psi}_{E_{N,l}}(q, \hbar), E_{N,l}(\hbar)]$ of the Dirac operator corresponding to the family $\Lambda^2(\omega)$ of type (4.3)–(4.5). Since $\Lambda^2(\omega)$ contains focal points, then (see Introduction) to construct the asymptotic solution as a whole (including the caustic and the region of a 'shadow') it is first necessary to find the asymptotic modulo $O(\hbar^{3/2})$ solutions $\Psi_E^j(q, \hbar)$ of (1.3) in each region $\pi_q(\Omega_j)$ of the configuration space $\mathbb{R}_q^3$. Subsection 5.1 *is devoted to solution of this problem.* Below *we restrict ourselves to considering* only the leading terms $\overset{\circ}{\Psi}{}_E^j(q, \hbar)$ of the local asymptotics $\Psi_E^j(q, \hbar)$ from which a multivalued function $\Psi_E(q, \hbar)$ (defined everywhere in $\mathbb{R}_q^3$) is matched in subsection 5.2. As a result, we construct a canonical operator with complex phase on the family $\Lambda^2(\omega)$. For a canonical operator to define a single-valued function $\overset{\circ}{\Psi}_E(q, \hbar)$ it is necessary to impose some additional conditions. These conditions lead to quasi-classical quantization conditions for the family $\Lambda^2(\omega)$ and are considered in subsection 5.3. A series of asymptotic eigenfunctions $\overset{\circ}{\Psi}_{E_{N,l}}(q, \hbar)$ constructed in this way satisfies condition (1.5) and it is made up of the functions localized at $\hbar \to 0$ in the domain of 'light' $\pi_q(\Lambda^2(\omega))$.

5.1. Local asymptotics of the Dirac operator

We begin with a review of some facts of the theory of the Weyl-ordered $\hbar^{-1}$-(pseudo)differential operators. Let $\hat{L} = L(-i\hbar\partial_q, q, \hbar)$ be a Weyl-ordered $\hbar^{-1}$-(pseudo)differential operator in the q-representation possessing a symbol $L(p, q, \hbar)$, then the action of the $\hat{L}$ operator on the function $\varphi(q)$ is defined by the formula

$$\hat{L}\varphi(q) = \frac{1}{(2\pi\hbar)^n} \int\int \mathrm{d}\xi\,\mathrm{d}p \exp\left[\frac{\mathrm{i}}{\hbar}\langle p, (q - \xi)\rangle\right] L\left(p, \frac{q+\xi}{2}, \hbar\right) \varphi(\xi) \tag{5.1.1}$$

where n is a dimension of the vector q. Introduce a direct and inverse $\hbar^{-1}$-Fourier transformation with respect to variables $q_{\bar{I}}$

$$F^{\hbar^{-1}}_{q_{\bar{I}} \to p_{\bar{I}}} \varphi(q) = \frac{1}{(2\pi \mathrm{i}\hbar)^{k/2}} \int \mathrm{d}q_{\bar{I}}\, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\langle p_{\bar{I}}, q_{\bar{I}}\rangle} \varphi(q) = \tilde{\varphi}(q_I, p_{\bar{I}}) \tag{5.1.2}$$

$$F^{-\hbar^{-1}}_{p_{\bar{I}} \to q_{\bar{I}}} \tilde{\varphi}(q_I, p_{\bar{I}}) = \frac{1}{(-2\pi \mathrm{i}\hbar)^{k/2}} \int \mathrm{d}p_{\bar{I}}\, \mathrm{e}^{+\frac{\mathrm{i}}{\hbar}\langle p_{\bar{I}}, q_{\bar{I}}\rangle} \varphi(q_I, p_{\bar{I}}) = \varphi(q) \tag{5.1.3}$$

where k is the number of components of the set $\bar{I}$. Then the Weyl-ordered operator $\hat{\tilde{L}} = \tilde{L}(-\mathrm{i}\hbar\partial_y, y, \hbar)$ in a mixed y-representation, where $y = (q_I, p_{\bar{I}})$ is given by

$$\tilde{L}(-\mathrm{i}\hbar\partial_y, y, \hbar) = F^{\hbar^{-1}}_{q_{\bar{I}}\to p_{\bar{I}}} L(-\mathrm{i}\hbar\partial_q, q, \hbar) F^{-\hbar^{-1}}_{p_{\bar{I}}\to q_{\bar{I}}}. \tag{5.1.4}$$

Allowing for equations (5.1.1)–(5.1.3) it is not difficult to show that the operator $\hat{\tilde{L}}$ in the y-representation may be obtained from the operator L in the q-representation by a formal substitution $-\mathrm{i}\hbar\partial_{q_{\bar{I}}} \to p_{\bar{I}}, q_{\bar{I}} \to \mathrm{i}\hbar\partial_{p_{\bar{I}}}$, i.e.

$$\tilde{L}(-\mathrm{i}\hbar\partial_y, y, \hbar) = L(-\mathrm{i}\hbar\partial_{q_I}, p_{\bar{I}}, q_I, \mathrm{i}\hbar\partial_{p_{\bar{I}}}). \tag{5.1.5}$$

Transition into a mixed y-representation enables one to construct a class of asymptotic solutions of (1.1) in the form of the $\hbar^{-1}$-Fourier transformation with respect to part variables

$$\Psi^{(I)}_E(q, \hbar) = F^{-\hbar^{-1}}_{p_{\bar{I}}\to q_{\bar{I}}} \tilde{\Psi}_E(y, \hbar) \tag{5.1.6}$$

where the function $\tilde{\Psi}_E(y, \hbar)$ is an asymptotic modulo $\mathrm{O}(\hbar^{3/2})$ solution of (1.1) in the y-representation

$$[\tilde{H}_{\mathrm{D}}(-\mathrm{i}\hbar\partial_y, y, \hbar) - E]\tilde{\Psi}_E(y, \hbar) = \mathrm{O}(\hbar^{3/2}). \tag{5.1.7}$$

Later on, function (5.1.6) describes a local asymptotic solution of (1.1) in the region of the configuration space $\mathbb{R}^3_q$ onto which the neighbourhood $\Omega_I \subset \Lambda^2$ is projected. The Weyl-ordered operator $\hat{\tilde{H}}_{\mathrm{D}}$ in (5.1.7) is obtained from the operator $\hat{H}_{\mathrm{D}}$ according to rule (5.1.5). We analyse the operator $\hat{H}_{\mathrm{D}}$ in some detail.

Let Cartesian and curvilinear coordinates of configuration space $\mathbb{R}^3_q$ be denoted by $x = (x^{\bar{a}})$, $q = (q^a)$, $\bar{a}, a = 1, 2, 3$, respectively. Introduce three vectors $e_a = (e^{\bar{a}}_a, \bar{a} = 1, 2, 3)$ with components $e^{\bar{a}}_a = \partial x^{\bar{a}}/\partial q^a$. Then, the Weyl-ordered Dirac operator $\hat{H}_{\mathrm{D}}$ may be written in the q-representation as

$$\hat{H}_{\mathrm{D}} = \hat{H}_0 + \hbar\hat{H}_1 \tag{5.1.8}$$

$$\hat{H}_0 = -\frac{c}{2}\alpha^{\bar{a}}(e^a_{\bar{a}}\hat{P}_a + \hat{P}_a e^a_{\bar{a}}) + \rho_3 mc^2 + eA_0 \tag{5.1.9}$$

$$\hat{H}_1 = \frac{\mathrm{i}c}{2}\alpha^{\bar{a}} e^a_{\bar{a},a'} \tag{5.1.10}$$

where ρ_3 and $\vec{\alpha} = (\alpha^{\bar{a}})$ are the Dirac matrices in the standard representation, $\hat{P}_a = \mathrm{i}\hbar\partial/\partial q^a - (e/c)A_a$, $e = -e_0$ is the electron charge, (A_0, A_a) are potentials of an external electromagnetic field.

The main symbol of operator (5.1.8) is the Hermitian matrix of the form

$$\overset{\circ}{H}(p, q) = c\vec{\alpha}\boldsymbol{P} + \rho_3 mc^2 + eA_0 \tag{5.1.11}$$

where $\boldsymbol{P} = (P_{\bar{a}})$ and $P_{\bar{a}} = e^a_{\bar{a}}(p_a + \frac{e}{c}A_a)$. Matrix (5.1.11), as pointed out in the introduction, has two doubly degenerate eigenvalues

$$\lambda^{(\pm)}(p, q) = eA_0 \pm \varepsilon \qquad \varepsilon = \sqrt{c^2\boldsymbol{P}^2 + m^2c^4}. \tag{5.1.12}$$

The eigenvectors corresponding to these can be combined into two 4×2-matrices $\Pi_\pm(p,q)$ which, at solutions $p=p(\tau)$, $q=q(\tau)$ of the Hamiltonian system (1.2), take the form

$$
\begin{aligned}
\Pi_+(\tau) &= \frac{1}{\sqrt{2(1+\gamma^{-1})}}\begin{pmatrix} 1+\gamma^{-1} \\ \sigma\beta \end{pmatrix} \\
\Pi_-(\tau) &= \frac{1}{\sqrt{2(1+\gamma^{-1})}}\begin{pmatrix} \sigma\beta \\ -(1+\gamma^{-1}) \end{pmatrix}
\end{aligned}
\tag{5.1.13}
$$

where $\gamma = \varepsilon/mc^2$, $\beta = \frac{1}{c}\dot{q} = \frac{1}{c}e_a\dot{q}^a$. Matrices (5.1.13) satisfy the orthonormality and completeness relations

$$
\Pi_\xi^+\Pi_\xi = \delta_{\xi'\xi} \qquad \sum_\xi \Pi_\xi\Pi_\xi^+ = 1 \qquad \xi = \pm 1. \tag{5.1.14}
$$

In addition, it is not difficult to prove the following identities:

$$
\begin{aligned}
&\langle\vec{\alpha}, m\rangle\Pi_\pm = \pm\Pi_\pm\langle\beta, m\rangle + \Pi_\mp\langle d, m\rangle \\
&d = \langle\sigma,\beta\rangle\frac{1}{1+\gamma^{-1}} - \sigma
\end{aligned}
\tag{5.1.15}
$$

where m is an arbitrary three-component vector.

$$
\begin{aligned}
&\frac{\partial}{\partial\tau_k}\Pi_\pm = \frac{1}{2}\Pi_\pm\frac{\langle\sigma,\beta\times\beta_{,\tau_k}\rangle}{1+\gamma^{-1}} \mp \frac{1}{2}\Pi_\mp\left\langle\sigma,\left(\beta\frac{\gamma\langle\beta,\beta_{,\tau_k}\rangle}{1+\gamma^{-1}} + \beta_{,\tau_k}\right)\right\rangle. \\
&k = 1,2.
\end{aligned}
\tag{5.1.16}
$$

We now proceed to construct asymptotic solutions of (5.1.7). The solution is searched for in the class of functions

$$
\tilde{\Psi}_E(y,\hbar) = \tilde{\Psi}_E(y,\tau_1(y),\tau_2(y),\hbar) \tag{5.1.17}
$$

in the form of the following asymptotic expansion in powers of $\hbar$

$$
\tilde{\Psi}_E(y,\hbar) = \sum_{n=0}^{2}\hbar^{n/2}(\Pi_+(\tau)\,\overset{n}{\tilde{J}}{}_E^{(+)}(y,\hbar) + \Pi_-(\tau)\,\overset{n}{\tilde{J}}{}_E^{(-)}(y,\hbar)) \tag{5.1.18}
$$

where the functions $\tau_1(y)$ and $\tau_2(y)$ were defined in (3.2). Denote by $Y_\hbar^{\bar{s}}$ the space of *the square-integrable scalar functions for which the following asymptotic estimations are* fulfilled on $\mathbb{R}_y^3$

$$
\begin{aligned}
&\Delta y = \hat{O}(\hbar^{1/2}) \qquad \Delta\hat{p}_y = \hat{O}(\hbar^{1/2}) \\
&(-i\hbar\partial_{\tau_k} + \hat{\tilde{a}}_k) = \hat{O}(\hbar) \qquad k = 1,2 \\
&Y_\hbar^{\bar{s}} = \left\{N_0(\hbar)\exp\left(\frac{i}{\hbar}\tilde{S}(y,\tau)\right)\sum_{|\kappa|=1}^{N}C_\kappa(\tau)\left(\frac{\Delta y}{\sqrt{\hbar}}\right)^\kappa \qquad N = 0,1,\ldots\right\} \\
&\kappa\in\mathbb{Z}_+^3 \qquad |\kappa| = \kappa_1+\kappa_2+\kappa_3
\end{aligned}
\tag{5.1.19}
$$

where $\tilde{S}(y,\tau)$ is defined in (3.4). The equality $\hat{F} = \hat{O}(\hbar^a)$, $a > 0$, implies that for the operator $\hat{F}$ in the space $Y_\hbar^{\tilde{s}}$ the following condition folds true: $\|\hat{F}\varphi\|_{L_2}/\|\varphi\|_{L_2} = O(\hbar^a)\ \forall\,\varphi \in Y_\hbar^{\tilde{s}}$, where $\|\cdot\|_{L_2} = \sqrt{\langle\cdot\,|\,\cdot\rangle_{L_2}}$. Then it is assumed that the two-component spinors $\tilde{J}_E^{(\pm)}$ in (5.1.18) may be represented as $\tilde{J}_E^{(\pm)}(y,\hbar) = v(\tau)\varphi(y,\hbar)$, where $\varphi(y,\hbar) \in Y_\hbar^{\tilde{s}}$ and the spinors $v(\tau)$ $(v^+(\tau)v(\tau) = 1)$ are to be determined.

Introduce the operator $\Delta\hat{\tilde{p}}_y = -i\hbar\partial_y - p_y(\tau)$. In the class of functions (5.1.17)–(5.1.19), the action of the operator $\Delta\hat{\tilde{p}}_y$ can be conveniently represented in the form

$$\Delta\hat{\tilde{p}}_y = \Delta\hat{p}_y + \sum_{k=1}^{2}(\partial_y\tau_k)(-i\hbar\partial_{\tau_k} + \hat{\tilde{a}}_k) - \sum_{k=1}^{2}(\partial_y\tau_k)\hat{\tilde{a}}_k \tag{5.1.20}$$

which, allowing for (3.5) and (5.1.19), enables one to give an asymptotic estimation for $\Delta\hat{\tilde{p}}_y$. Now we expand the operator $\hat{\tilde{H}}_{\mathrm{D}}$ in a neighbourhood of the manifold $\Lambda^2 = \{(p,q) : p = p(\tau), q = q(\tau)\}$ in the Taylor power series over the operator $\Delta\hat{\tilde{p}}_y$ and Δy up to second-order terms. Then, equation (5.1.7) turns out to be equivalent to the set of conditions

$$\left[-i\hbar\sum_{k=1}^{2}\langle\mathring{H}_{p_y}(\tau),\partial_y\tau_k\rangle\frac{\partial}{\partial\tau_k} + \mathring{H}(\tau) + \tilde{\Delta}\mathring{H}(\tau) - E\right]\tilde{\Psi}_E(y,\hbar) = O(\hbar^{3/2}) \tag{5.1.21}$$

$$\hat{\tilde{a}}_k\tilde{\Psi}_E(y,\hbar) = O(\hbar) \qquad k = 1,2. \tag{5.1.22}$$

Here and below $\tilde{\Delta} = \hat{\tilde{\delta}}^1 + \frac{1}{2}\hat{\tilde{\delta}}^2$, and the expression $\hat{\tilde{\delta}}^k\mathring{L}(\tau)$, $k = 1,2$, implies the kth term in the Taylor power expansion over the operators $\Delta\hat{\tilde{p}}_y$ and Δy of the Weyl-ordered operator $\hat{\tilde{L}} = \tilde{L}(-i\hbar\partial_y, y, \hbar)$ with the main symbol $\tilde{L}(p_y, y) = \mathring{L}(p,q)$ in a neighbourhood of Λ^2. The spectral parameter E in (5.1.21) is given as

$$E = E_0 + \hbar E_1 + O(\hbar^2) \tag{5.1.23}$$

where $E_0 = \lambda^{(+)}|_{\Lambda^2(\omega)}$. Substitute (5.1.18) and (5.1.23) into (5.1.21) and consider the expression at $\Pi_-(\tau)$. By combining the terms of order $\hbar^0$, $\hbar^{1/2}$, and $\hbar$, respectively, we obtain the following chain of conditions:

$$\overset{0}{\tilde{J}}{}_E^{(-)} = 0 \qquad \overset{1}{\tilde{J}}{}_E^{(-)} = \frac{1}{2\varepsilon}\hat{\tilde{Q}}_1\overset{0}{\tilde{J}}{}_E^{(+)} \qquad \overset{2}{\tilde{J}}{}_E^{(-)} = \frac{1}{2\varepsilon}(\hat{\tilde{Q}}_1\overset{1}{\tilde{J}}{}_E^{(+)} + \hat{\tilde{Q}}_2\overset{0}{\tilde{J}}{}_E^{(+)}) \tag{5.1.24}$$

where the operators $\hat{\tilde{Q}}_1$ and $\hat{\tilde{Q}}_2$ are of the form

$$\hbar^{1/2}\hat{\tilde{Q}}_1 = -\Pi_-^{\pm}\sum_{k=1}^{2}\langle\mathring{H}_{p_y}(\tau),\partial_y\tau_k\rangle\Pi_+\hat{\tilde{a}}_k + c\langle\boldsymbol{d},\hat{\tilde{\delta}}^1\boldsymbol{P}\rangle \tag{5.1.25}$$

$$\begin{aligned}\hbar\hat{\tilde{Q}}_2 = \frac{\hbar^{1/2}}{2\varepsilon}&\left(-\Pi_-^{\pm}\sum_{k=1}^{2}\langle\mathring{H}_{p_y}(\tau),\partial_y\tau_k\rangle\Pi_-\hat{\tilde{a}}_k - \langle\dot{q},\hat{\tilde{\delta}}^1\boldsymbol{P}\rangle + e\hat{\tilde{\delta}}^1A_0\right)\hat{\tilde{Q}}_1\\ &+\Pi_-^{\pm}\sum_{k=1}^{2}\langle\mathring{H}_{p_y}(\tau),\partial_y\tau_k\rangle(-i\hbar\partial_{\tau_k} + \hat{\tilde{a}}_k)\Pi_+ + \frac{c}{2}\langle\boldsymbol{d},\hat{\tilde{\delta}}^2\boldsymbol{P}\rangle + \hbar\Pi_-^{\pm}H_1(\tau)\Pi_+.\end{aligned} \tag{5.1.26}$$

Additionally, the condition

$$\hat{\hat{a}}_k \overset{\circ}{\tilde{J}}{}_E^{(+)}(y,\hbar) = 0 \qquad k = 1,2 \tag{5.1.27}$$

is assumed to be fulfilled. Then allowing for (5.1.24)–(5.1.27) a similar treatment of the expression at $\Pi_+(\tau)$ in the left-hand side of (5.1.21) results in the equation for the spinor $\overset{\circ}{\tilde{J}}{}_E^{(+)}(y,\hbar)$:

$$\left[-i\hbar\Pi_+^\dagger \sum_{k=1}^{2} \langle \overset{\circ}{H}_{p_y}(\tau), \partial_y\tau_k\rangle \frac{\partial}{\partial\tau_k}\Pi_+ + \frac{1}{2}\langle \dot{q}, \hat{\hat{\delta}}^2 \boldsymbol{P}\rangle + \frac{e}{2}\hat{\hat{\delta}}^2 A_0 + \hbar\Pi_+^\dagger H_1(\tau)\Pi_+ - \hbar E_1 \right.$$
$$\left. + \frac{c}{2\varepsilon}\left(+ \Pi_+^\dagger \sum_{k=1}^{2} \langle \overset{\circ}{H}_{p_y}(\tau), \partial_y\tau_k\rangle \Pi_- \hat{\hat{a}}_k + c\langle \boldsymbol{d}, \hat{\hat{\delta}}^1 \boldsymbol{P}\rangle \right) \langle \boldsymbol{d}, \hat{\hat{\delta}}^1 \boldsymbol{P}\rangle \right] \overset{\circ}{\tilde{J}}{}_E^{(+)}(y,\hbar) = 0. \tag{5.1.28}$$

After further simplification, in view of (3.2), equation (5.1.28) takes the form of the Pauli-type equation:

$$\left[(-i\hbar\partial_{\tau_1} + \hat{\hat{\lambda}}_0^{(+)} - E) - i\hbar\frac{\partial}{\partial\tau_1}\ln g(\tau)^{1/4} + \hbar\langle \sigma, \boldsymbol{\mathcal{B}}(\tau)\rangle \right] \overset{\circ}{\tilde{J}}{}_E^{(+)}(y,\hbar) = 0. \tag{5.1.29}$$

Here, the operator $\hat{\hat{\lambda}}_0^{(+)}$ was defined in (3.10), $g = \det(\eta_{ab})$, where $\eta_{ab} = \langle \boldsymbol{e}_a, \boldsymbol{e}_b\rangle$ is a Cartesian metric on $\mathbb{R}^3_q$ in curvilinear coordinates (q^a); and $\boldsymbol{\mathcal{B}}(\tau) : \mathbb{R}^2_\tau \to \mathbb{R}^3$ is the 'polarization' vector equal to

$$\boldsymbol{\mathcal{B}}(\tau) = \frac{e_0 c}{2\varepsilon}\left(\boldsymbol{H}(\tau) - \frac{\boldsymbol{\beta}\times\boldsymbol{E}(\tau)}{1+\gamma^{-1}} \right). \tag{5.1.30}$$

In (5.1.30) $\boldsymbol{E}(\tau)$ and $\boldsymbol{H}(\tau)$ are the electric and magnetic components of an external electromagnetic field.

Assume that

$$\overset{\circ}{\tilde{J}}{}_E^{(+)}(y,\hbar) = g(\tau)^{-1/4}|\nu, \tau(y)\rangle v(\tau) \tag{5.1.31}$$

where $v(\tau)$ is a two-dimensional spinor which is to be defined. Then, taking into account (3.11), one obtains the following equation for $v(\tau)$

$$\left(-i\frac{\partial}{\partial\tau_1} + \langle \sigma, \boldsymbol{\mathcal{B}}(\tau)\rangle \right) v(\tau) = 0. \tag{5.1.32}$$

Finally, it should be noted that (5.1.27) follows from (3.9) and (5.1.31). Whence, allowing for (5.1.24), the validity of (5.1.22) is obtained. As a result, the task of constructing the asymptotic solutions $\tilde{\Psi}_E(y,\hbar)$ of (5.1.7) is reduced to solution of the ordinary linear differential equation (5.1.32) with respect to the variable τ_1 (the variable τ_2 is considered as a parameter) with a subsequent substitution of the functions $\tau = \tau(y)$ (3.2) into the solution obtained.

Now we write out an explicit form of functions (5.1.18). According to (5.1.24) we have

$$\tilde{\Psi}_E(y,\hbar) = \left[\Pi_+(\tau) + \frac{1}{2\varepsilon}\Pi_-(\tau)(\hbar^{1/2}\hat{\tilde{Q}}_1 + \hbar\hat{\tilde{Q}}_2)\right]\overset{\circ}{\tilde{J}}{}^{(+)}_E(y,\hbar)$$

$$+\hbar^{1/2}\left[\Pi_+(\tau) + \frac{\hbar^{1/2}}{2\varepsilon}\Pi_-(\tau)\hat{\tilde{Q}}_1\right]\overset{1}{\tilde{J}}{}^{(+)}_E(y,\hbar) + \hbar\Pi_+(\tau)\overset{2}{\tilde{J}}{}^{(+)}_E(y,\hbar). \quad (5.1.33)$$

Thus, solution (5.1.33) contains the two arbitrary spinors $\overset{n}{\tilde{J}}{}^{(+)}_E(y,\hbar)$, $n = 1, 2$, which can be found from the following $(\operatorname{mod} O(\hbar^{5/2}))$ approximation. According to (5.1.31) the leading term of asymptotic (5.1.33) is

$$\overset{\circ}{\tilde{\Psi}}_E(y,\hbar) = \{g(\tau)^{-1/4}\Pi_+(\tau)v(\tau)|\nu,\tau\rangle\}\big|_{\tau=\tau(y)}. \quad (5.1.34)$$

In conclusion it should be noted that if in case of scalar equations (e.g. the Schrödinger or Klein–Gordon equations) the asymptotic $(\operatorname{mod} O(\hbar^{3/2}))$ solution and its leading term coincide, then in the case of equations with a matrix Hamiltonian it does not occur, as is seen from (5.1.33) and (5.1.34). The fact that there are two arbitrary two-component spinors in the asymptotic $(\operatorname{mod} O(\hbar^{3/2}))$ solution (5.1.33) is not accidental and reflects one of the characteristic features inherent to the matrix equations [8]. This condition essentially complicates the matching procedure of the local asymptotics (5.1.6) in an attempt to obtain an asymptotic $(\operatorname{mod} O(\hbar^{3/2}))$ solution of (1.1) which is uniformity valid everywhere in $\mathbb{R}^3_q$. However, as will be seen below, to obtain the quasi-classical quantization conditions it is sufficient to restrict oneself to constructing only the leading terms (5.1.34).

5.2. Maslov canonical operator on the family of Lagrangian manifolds with complex germ corresponding to the Dirac operator in axially symmetric external field

According to the general Maslov theory [3, 6] in order to construct unified regular quasi-classical asymptotics (defined in the whole configuration space) it is necessary to build up a special operator which, in the case of incomplete-dimensional Lagrangian manifolds, is known as a canonical operator with complex phase.

Here, we shall restrict ourselves to a special class of invariant Lagrangian manifolds $\Lambda^2(\omega)$ introduced in section 4. In this case the procedure for constructing the canonical operator is essentially simplified and will be, in fact, reduced to the well known construction of the canonical operator with real phase on the closed curve $\Lambda^1(E_0, I_0)$ [3, 9]. We show below how this can be done.

Define a suitable partition of unity submitted to the coverage $\{V_j\}$, $j = \overline{1,4}$ of the closed curve $\Lambda^1(\omega)$, $\omega = (E_0, I_0)$, i.e. a set of C^∞-functions $e_j(\tau_1)$ such that

$$\operatorname{supp} e_j(\tau_1) \subset V_j, \sum_{j=1}^{4} e_j(\tau_1) = 1 \qquad \text{for all } \tau_1.$$

The two arbitrary neighbourhoods V_{j_1} and V_{j_2} possessing a non-empty intersection are chosen from the set $\{V_j\}$. Let the neighbourhood V_{j_1} be singular and V_{j_2} non-singular. The Maslov index of the pair of neighbourhoods (V_{j_1}, V_{j_2}) is, by definition, the number

$$\gamma(V_{j_1}, V_{j_2}) = \left(\operatorname{inerdex}\frac{\dot{q}_1}{\dot{p}_1}(\tau_1)\right)\Bigg|_{\tau_1\in V_{j_1}\cap V_{j_2}} \quad (5.2.1)$$

where, in addition, an assumption is made that $\gamma(V_{j_1}, V_{j_2}) = -\gamma(V_{j_2}, V_{j_1})$. Let $l(\mathring{\tau}_1, \tau_1)$ denote the path on $\Lambda^1(\omega)$ oriented counterclockwise along which the integration is carried out in (4.15) and (4.17). Let $V_1, \ldots, V_j$ be an ordered chain of neighbourhoods in which the path $l(\mathring{\tau}_1, \tau_1)$ lies with $\mathring{\tau}_1 \in V_1$, $\tau_1 \in V_j$. Then, the value equal to

$$\gamma_j[l(\mathring{\tau}_1, \tau_1)] = \gamma(V_1, V_2) + \ldots + \gamma(V_{j-1}, V_j) \tag{5.2.2}$$

has the meaning of the Maslov index of the above chain.

Let $\mathcal{F}(\tau) \equiv \tilde{\mathcal{F}}(\varphi(\tau), \tau_1) : \mathbb{R}^2_\tau \to \mathbb{C}^4$ denote a smooth finite vector function where the function $\varphi(\tau)$ was defined in (4.8). Then, in view of the note at the end of section 4 we shall have

$$\mathcal{F}(\tau)\big|_{\tau=\tau^j(y)} = \tilde{\mathcal{F}}(\varphi, \tau_1)\big|_{\tau_1=\tau_1^j(y)}.$$

Introduce into consideration the local pre-canonical operator $K_{\Lambda^2(\omega)}(\Omega_j)$ which acts on the functions $\mathcal{F}(\tau)$:

(i) in the case of non-singular neighbourhoods Ω_{j_2}, $j_2 = 1, 3$

$$(K_{\Lambda^2(\omega)}(\Omega_{j_2})[\mathcal{F}(\tau)])(q) = \left(\mathrm{e}^{\frac{i\pi}{2}\gamma_{j_2}[l(\mathring{\tau}_1,\tau_1)]}|\nu, \tau_1\rangle \tilde{\mathcal{F}}(\varphi, \tau_1)\right)\Bigg|_{\tau_1=\tau_1^{j_2}(q_1)} \tag{5.2.3}$$

(ii) in the case of singular neighbourhoods Ω_{j_1}, $j_1 = 2, 4$

$$(K_{\Lambda^2(\omega)}(\Omega_{j_1})[\mathcal{F}(\tau)])(q) = \frac{1}{\sqrt{-2\pi i\hbar}} \int \mathrm{d}p_1 \left(\mathrm{e}^{\frac{i\pi}{2}\gamma_{j_1}[l(\mathring{\tau}_1,\tau_1)]}\mathrm{e}^{\frac{i}{\hbar}p_1q_1}|\nu, \tau_1\rangle \tilde{\mathcal{F}}(\varphi, \tau_1)\right)\Bigg|_{\tau_1=\tau_1^{j_1}(p_1)} \tag{5.2.4}$$

where the functions $|\nu, \tau_1^{j_2}(q_1)\rangle$ and $|\nu, \tau_1^{j_1}(p_1)\rangle$ were defined in (4.15) and (4.17), respectively. The pre-canonical operators introduced in this way satisfy the following important property: for any pair of the neighbourhoods Ω_{j_1} and Ω_{j_2} in their overlap region the following equality is valid:

$$(K_{\Lambda^2(\omega)}(\Omega_{j_2})[\mathcal{F}(\tau)])(q) = (K_{\Lambda^2(\omega)}(\Omega_{j_1})[\mathcal{F}(\tau)])(q) + \mathrm{O}(\hbar) \qquad q \in \pi_q\Omega_{j_1} \cap \pi_q\Omega_{j_2}. \tag{5.2.5}$$

The proof of this statement is based on the stationary phase method and given in appendix 1.

By means of the operators $K_{\Lambda^2(\omega)}(\Omega_j)$ and a partition of unity $e_j(\tau_1)$ submitted to the neighbourhoods V_j, $j = \overline{1,4}$, we shall build up the operator $K_{\Lambda^2(\omega)}$ whose action on the functions $\mathcal{F}(\tau)$ is defined by the rule

$$(K_{\Lambda^2(\omega)}[\mathcal{F}(\tau)])(q) = \sum_j (K_{\Lambda^2(\omega)}(\Omega_j)[e_j(\tau_1)\mathcal{F}(\tau)])(q). \tag{5.2.6}$$

The operator $K_{\Lambda^2(\omega)}$ is called the canonical operator on the family $[\Lambda^2(\omega), r^3(\Lambda^2(\omega))]$. In view of (5.2.5) it can be shown that the canonical operator with an accuracy of order $\mathrm{O}(\hbar)$, first, does not depend on the manner of partition of unity $\{e_j\}$ with the fixed atlas $\{\Omega_j\}$, and, secondly, does not depend on the choice of the canonical atlas itself on $\Lambda^2(\omega)$.

For further convenience we shall introduce a set of functions of the form

$$\Phi_\nu(\tau_1, q_2, \varphi) = \left(\exp\frac{\mathrm{i}}{\hbar}\left[I_0\varphi + (E - E_0)\tau_1 + \frac{1}{2}\operatorname{Re}\frac{w(\tau_1)}{z(\tau_1)}q_2^2\right]\right)\Xi_\nu(q_2, \tau_1)\bar{\mathcal{F}}(\varphi, \tau_1) \quad (5.2.7)$$

where $\Xi_\nu(q_2, \tau_1)$ were defined in (4.16). Then, relation (5.2.6) may be rewritten as follows:

$$(K_{\Lambda^2(\omega)}[\mathcal{F}(\tau)])(q) \equiv (K_{\Lambda^1(\omega)}[\Phi_\nu(\tau_1, q_2, \varphi)])(q) = \sum_j (K_{\Lambda^1(\omega)}(V_j)[\Phi_\nu(\tau_1, q_2, \varphi)])(q)$$

$$= \sum_{j_2}\left[\frac{\exp\left(\mathrm{i}\frac{\pi}{2}\gamma_{j_2}\right)e_{j_2}(\tau_1)}{|\dot{q}_1(\tau_1)|^{1/2}}\exp\left(\frac{\mathrm{i}}{\hbar}\int_{\overset{\circ}{\tau}_1}^{\tau_1} p_1(t)\,\mathrm{d}q_1(t)\right)\Phi_\nu(\tau_1, q_2, \varphi)\right]\Bigg|_{\tau_1=\tau_1^{j_2}(q_1)}$$

$$+ \sum_{j_1}\frac{1}{\sqrt{-2\pi\mathrm{i}}}\int \mathrm{d}p_1\left\{\frac{\exp\left(\mathrm{i}\frac{\pi}{2}\gamma_{j_1}\right)e_{j_1}(\tau_1)}{|\dot{p}_1(\tau_1)|^{1/2}}\right.$$

$$\left.\times\exp\left[\frac{\mathrm{i}}{\hbar}\left(\int_{\overset{\circ}{\tau}_1}^{\tau_1} p_1(t)\,\mathrm{d}q_1(t) + p_1\Delta q_1\right)\right]\Phi_\nu(\tau_1, q_2, \varphi)\right\}\Bigg|_{\tau_1=\tau_1^{j_1}(p_1)}. \quad (5.2.8)$$

Note. It should be emphasized that in (5.2.8) the parameter τ_1 takes on the values on the section $\overset{\circ}{\tau}_1 \leqslant \tau_1 \leqslant \overset{\circ}{\tau}_1 + T_1$. Whence, the following restriction on the choice of the functions $\tau_1^{j_2}(q_1)$ and $\tau_1^{j_1}(p_1)$ is obtained: $\overset{\circ}{\tau}_1 \leqslant \tau_1^{j_1}(p_1) \leqslant \overset{\circ}{\tau}_1 + T_1$, $\overset{\circ}{\tau}_1 \leqslant \tau_1^{j_2}(q_1) \leqslant \overset{\circ}{\tau}_1 + T_1$.

It is curious to note that in (5.2.8) the construction of the Maslov canonical operator with real phase $K_{\Lambda^1(\omega)}$ on the family of closed curves $\Lambda^1(\omega)$ is explicity present [3, 9]. This fact is likely to be due to a special choice of the class of invariant Lagrangian tori $\Lambda^2(\omega)$.

Now, if we avail ourselves of the familier property of the δ-function: $\delta(x - x(t)) = \delta(t - t(x))/|\dot{x}(t(x))|$, where the function $t(x)$ obeys the condition $x(t(x)) = x$ and take that

$$\arg\dot{q}_1(\tau_1) = \pi\gamma_{j_2}[l(\overset{\circ}{\tau}_1, \tau_1)] \quad (5.2.9)$$

$$\arg\dot{p}_1(\tau_1) = \pi\gamma_{j_1}[l(\overset{\circ}{\tau}_1, \tau_1)] \quad (5.2.10)$$

then, it is not difficult to obtain the following equivalent representation for (5.2.8):

$$(K_{\Lambda^2(\omega)}[\mathcal{F}(\tau)])(q) = \int_{\overset{\circ}{\tau}_1}^{\overset{\circ}{\tau}_1+T_1}\mathrm{d}\tau_1\,\psi(q, \tau_1, \hbar)$$

$$\equiv \int_{\overset{\circ}{\tau}_1}^{\overset{\circ}{\tau}_1+T_1}\mathrm{d}\tau_1\left\{(e_1(\tau_1) + e_3(\tau_1))\sqrt{\dot{q}_1(\tau_1)}\exp\left(\frac{\mathrm{i}}{\hbar}\int_{\overset{\circ}{\tau}_1}^{\tau_1} p_1(t)\,\mathrm{d}q_1(t)\right)\right.$$

$$\times\,\delta(q_1 - q_1(\tau_1)) + \frac{e_2(\tau_1) + e_4(\tau_1)}{\sqrt{-2\pi\mathrm{i}\hbar}}\sqrt{\dot{p}_1(\tau_1)}$$

$$\left.\times\exp\left(\frac{\mathrm{i}}{\hbar}\left[\int_{\overset{\circ}{\tau}_1}^{\tau_1} p_1(t)\,\mathrm{d}q_1(t) + p_1(\tau_1)(q_1 - q_1(\tau_1))\right]\right)\right\}\Phi_\nu(\tau_1, q_2, \varphi).$$

$$(5.2.11)$$

Consider the action of the canonical operator $K_{\Lambda^2(\omega)}$ (5.2.11) on the function $\mathcal{F}(\tau) = (g(\tau_1))^{-1/4}\Pi_+(\tau)v(\tau)$, where $\Pi_+(\tau)$ *and* $v(\tau)$ *were defined in* (5.1.13) *and* (5.1.32), respectively. Assume in this case

$$\overset{\circ}{\Psi}_E(q, \hbar) = \left(K_{\Lambda^2(\omega)}\left[\frac{\Pi_+(\tau)v(\tau)}{(g(\tau_1))^{1/4}}\right]\right)(q, \hbar) = \int_{\overset{\circ}{\tau}_1}^{\overset{\circ}{\tau}_1+T_1}\mathrm{d}\tau_1\,\overset{\circ}{\psi}_E(q, \tau_1, \hbar). \quad (5.2.12)$$

Then, for function (5.1.12) the following properties are true:

(1) The function $\overset{\circ}{\Psi}_E(q,\hbar)$ is localized in the classically allowed region, i.e. in the domain of the projection of $\Lambda^2(\omega)$ onto configuration space $\mathbb{R}^3_q$. In particular, it decays exponentially with respect to the variable q_2 and has the form of the Gaussian wavepacket centred at the point $q_2 = 0$. The asymptotic behaviour of the function $\overset{\circ}{\Psi}_E(q,\hbar)$ near the caustic and in the region of 'shadow' is presented in appendix 2.

(2) At the points $q \in \pi_q(\Lambda^2(\omega))$ (the region of 'light') the function $\overset{\circ}{\Psi}_E(q,\hbar)$ behaves as follows.

(i) If $q = \{(q_1(\tau_1), q_2, \varphi), \tau_1 \in \operatorname{supp} e_{j_2}, \tau_1 \notin \operatorname{supp} e_{j_1}\}$, then the function $\overset{\circ}{\Psi}_E(q,\hbar)$ is the combination of the functions (5.2.3) and, according to (5.1.6) and (5.1.34), describes the leading term of the local asymptotic solution of the Dirac equation in the region of the configuration space onto which the non-singular neighbourhoods Ω_{j_2} are projected.

(ii) If $q = \{(q_1(\tau_1), q_2, \varphi), \tau_1 \in \operatorname{supp} e_{j_1}, \tau_1 \notin \operatorname{supp} e_{j_2}\}$, then the functions $\overset{\circ}{\Psi}_E(q,\hbar)$ is a combination of the functions (5.2.4) and describes the leading term of the local asymptotic solution in the region of the projection $\pi_q(\Omega_{j_1})$ of the focal neighbourhoods Ω_{j_1}.

(iii) If $q \in \pi_q(\Omega_{j_1}) \cap \pi_q(\Omega_{j_2})$, then the function $\overset{\circ}{\Psi}_E(q,\hbar)$ contains, at first sight, both types of functions (5.2.3) and (5.2.4). However, according to (5.2.5), they merge into each other (at least with an accuracy to order $O(\hbar)$) in the transition region where Ω_{j_1} and Ω_{j_2} overlap. If one allows for the fact that

$$\sum_{j=1}^{4} e_j(\tau_1) = 1 \qquad \text{for all } \tau_1$$

the canonical operator $K_{\Lambda^2(\omega)}$ ensures a correct matching of the local asymptotics.

Note. Independence of the metric determinant $g = \det(\eta_{ab})$ on the parameter τ_2 in (5.2.12) follows from the cyclicity condition of the Hamiltonian function (5.1.12) with respect to the variable φ and from (4.8).

5.3. Quasi-classical quantization conditions of family $[\Lambda^2(\omega), r^3(\Lambda^2(\omega))]$

Formula (5.2.12) obtained in the previous subsection for the function $\overset{\circ}{\Psi}_E(q,\hbar)$, and which gives a locally matched asymptotic solution of the Dirac equation, does not depend (with an accuracy to order $O(\hbar)$) on the choice of the canonical atlas on $\Lambda^2(\omega)$, the choice of the representation in the region of the neighbourhoods intersection, or the partition of unity. However, as is seen from (5.2.11), it depends on the choice of the initial point $\overset{\circ}{\tau}_1$ on the closed curve $\Lambda^1(\omega)$. It is natural to require independence of construction (5.2.12) on the choice of the 'time origin' $\overset{\circ}{\tau}_1$, which is automatically fulfilled on condition of T_1-periodicity of the function $\overset{\circ}{\Psi}_E(q,\tau_1,\hbar)$ with respect to the variable τ_1:

$$\overset{\circ}{\Psi}_E(q,\tau_1 + T_1,\hbar) = \overset{\circ}{\Psi}_E(q,\tau_1,\hbar). \tag{5.3.1}$$

Also, the function $\overset{\circ}{\Psi}_E(q,\tau_1,\hbar)$ should satisfy the 2π-periodicity condition with respect to the variable $\varphi \bmod 2\pi$

$$\overset{\circ}{\Psi}_E(q_1,q_2,\varphi + 2\pi,\tau_1,\hbar) = \overset{\circ}{\Psi}_E(q,\tau_1,\hbar). \tag{5.3.2}$$

Under these conditions the canonical operator $K_{\Lambda^2(\omega)}$ defines the single-valued function $\overset{\circ}{\Psi}_E(q,\hbar)$. In addition, from (5.3.1) and (5.3.2) the quasi-classical quantization conditions of the parameters $\omega = (E_0, I_0)$ of the family $[\Lambda^2(\omega), r^3(\Lambda^2(\omega))]$ result.

The following commentary proceeds to derive these conditions.

As follows from (4.8) and (5.1.30), the dependence of the polarization vector $\mathcal{B}(\tau)$ on the arguments τ_1 and τ_2 is generally such that $\mathcal{B}(\tau) = \tilde{\mathcal{B}}(\varphi(\tau), \tau_1)$. Also, the 2π- and T_1-periodicity conditions are fulfilled: $\tilde{\mathcal{B}}(\varphi(\tau)+2\pi, \tau_1) = \tilde{\mathcal{B}}(\varphi(\tau), \tau_1)$, $\tilde{\mathcal{B}}(\varphi(\tau), \tau_1 + T_1) = \tilde{\mathcal{B}}(\varphi(\tau), \tau_1)$. Therefore, the vector function $\tilde{\mathcal{B}}(\varphi(\tau), \tau_1)$ behaves as a doubly periodic function with respect to the variables $\varphi(\tau)$ and τ_1. Later on it will be assumed that (5.1.32) permits a set of two linearly independent Floquet solutions $v_\zeta(\tau) = \tilde{v}_\zeta(\varphi(\tau), \tau_1)$, $\zeta = \pm 1$, such that

$$\tilde{v}_\zeta(\varphi(\tau) + 2\pi, \tau_1) = \tilde{v}_\zeta(\varphi(\tau), \tau_1) \tag{5.3.3}$$

$$\tilde{v}_\zeta(\varphi(\tau), \tau_1 + T_1) = \mathrm{e}^{-\mathrm{i}\omega_\zeta^s T_1} \tilde{v}_\zeta(\varphi(\tau), \tau_1) \qquad \operatorname{Im} \omega_\zeta^s = 0, \tag{5.3.4}$$

$$\overset{+}{v}_{\zeta'} v_\zeta = \delta_{\zeta\zeta'} \qquad \sum_\zeta v_\zeta \overset{+}{v}_\zeta = 1. \tag{5.3.5}$$

Note. The above assumption for the vector $\mathcal{B}(\tau)$ is also valid for the matrix $\Pi_+(\tau) = \tilde{\Pi}_+(\varphi(\tau), \tau_1)$.

The phase increment of the functions $\sqrt{\dot{q}_1(\tau_1)}$ and $\sqrt{\dot{p}_1(\tau_1)}$ are left to be found while going around the closed path $\Lambda^1(\omega)$ counterclockwise. According to (5.2.2) we have $\gamma_j[l(\overset{\circ}{\tau}_1, \tau_1 + T_1)] = \gamma_j[l(\overset{\circ}{\tau}_1, \tau_1)] + \gamma[\Lambda^1(\omega)]$. Since in our case $\gamma[\Lambda^1(\omega)] = 2$, from (5.2.9) and (5.2.10) it follows that

$$\sqrt{\dot{q}_1(\tau_1 + T_1)} = \mathrm{e}^{\mathrm{i}\pi}\sqrt{\dot{q}_1(\tau_1)} \qquad \sqrt{\dot{p}_1(\tau_1 + T_1)} = \mathrm{e}^{\mathrm{i}\pi}\sqrt{\dot{p}_1(\tau_1)}. \tag{5.3.6}$$

Now, it is not difficult to obtain the conditions when (5.3.1) and (5.3.2) are valid. Allowing for the explicit form of the function $\overset{\circ}{\psi}_E(q, \tau_1, \hbar)$ and (5.3.3), relationship (5.3.2) leads to quantization of the momentum integral I_0:

$$I_0 = \hbar l \qquad l = \pm 1, \pm 2, \ldots. \tag{5.3.7}$$

Making use of (4.14), (5.3.4) and (5.3.6), relationship (5.3.1) results in the following condition

$$\int_0^{T_1} p_1(t, E_0, I_0)\dot{q}_1(t, E_0, I_0)\,\mathrm{d}t + \hbar T_1\{E_1 - \Omega_1(\nu + \tfrac{1}{2}) - \omega_\zeta^s\} = 2\pi\hbar(n + \tfrac{1}{2}) + \mathrm{O}(\hbar^2) \tag{5.3.8}$$

where $n = \pm 1, \pm 2, \ldots$, $\nu = 0, 1, 2, \ldots$, $\zeta = \pm 1$. Equation (5.3.8) obviously holds, if the parameters E_0 and E_1 are defined from the conditions

$$\frac{1}{2\pi}\int_0^{T_1} p_1(t, E_0, I_0)\dot{q}_1(t, E_0, I_0)\,\mathrm{d}t = \hbar(n + \tfrac{1}{2}) \tag{5.3.9}$$

$$E_1 = \Omega_1(\nu + \tfrac{1}{2}) + \omega_\zeta^s. \tag{5.3.10}$$

From (5.3.9) and (5.3.10) together with (5.3.4) one obtains, with an accuracy to $O(\hbar^2)$, a spectral sequence of the energy levels

$$E_{N,l}(\hbar) = E_{n,l,\nu,\zeta}(\hbar) = E^{(0)}_{n,l}(\hbar) + \hbar E^{(1)}_{n,l,\nu,\zeta}(\hbar) + O(\hbar^2). \tag{5.3.11}$$

The quantum numbers $l = l(\hbar)$, $n = n(\hbar)$ and the parameter $\hbar$ should be tied by the conditions:

$$\lim_{\hbar\to 0} \hbar l(\hbar) = I_0^{\mathrm{cl}} \qquad \lim_{\hbar\to 0} \hbar n(\hbar) = \frac{1}{2\pi}\oint_{\Lambda^1(E_0^{\mathrm{cl}}, I_0^{\mathrm{cl}})} p_1\,\mathrm{d}q_1 .$$

In this case the series of eigenvalues (5.3.11) in the limit at $\hbar \to 0$ corresponds to a relativistic electron moving along the classical trajectory confined to the invariant torus $\Lambda^2(E_0^{\mathrm{cl}}, I_0^{\mathrm{cl}})$ with given values of energy E_0^{cl} and momentum I_0^{cl}.

It can be shown also that in expanding in $\hbar$ with an accuracy to $O(\hbar^2)$, conditions (5.3.9) and (5.3.10) are equivalent to the quantization condition of the spectral parameter E of the Bohr–Sommerfeld type:

$$\frac{1}{2\pi\hbar}\oint_{\Lambda^1(E, I=\hbar l(\hbar))} p_1(t, E, I)\,\mathrm{d}q_1(t, E, I) = \frac{T_1}{2\pi}\{\Omega_1(\nu + \tfrac{1}{2}) + \omega^s_\zeta\} + (n(\hbar) + \tfrac{1}{2}) \tag{5.3.12}$$

where $E = E_0 + \hbar E_1 + O(\hbar^2)$. Equation (5.3.12) represents the basic result obtained in this subsection.

We should point out the presence of the half-integer addition $\frac{1}{2}$ in the quantization rule (5.3.12). Its appearance is due to the occurrence of the non-trivial quantity $\gamma[\Lambda^1(\omega)] = \mathrm{ind}\,\Lambda^1(\omega)$—the Maslov index for the closed curve $\Lambda^1(\omega)$. It is well known that in the case of the family of complete-dimensional invariant tori Λ^n the quantization conditions contain the Maslov indices of oriented closed curves forming a basis of one-cycles on the manifold Λ^n [8, 9] (see also [2, 18, 19] interpreting the Maslov index). However, unlike the complete-dimensional case the Maslov index may be formally excluded from the quantization conditions of the family of incomplete-dimensional tori Λ^k, $k < n$. In support of this statement consider an example. In (5.3.12) the characteristic Floquet index Ω_1 and the main quantum number $n(\hbar)$ are defined in the following way: $\Omega_1 = \tilde{\Omega}_1 - 2\pi/T_1$, $n(\hbar) = \tilde{n}(\hbar) + \nu$. Then (5.3.12) is rewritten as

$$\frac{1}{2\pi\hbar}\oint_{\Lambda^1} p_1\,\mathrm{d}q_1 = \frac{T_1}{2\pi}\{\tilde{\Omega}_1(\nu + \tfrac{1}{2}) + \omega^s_\zeta\} + \tilde{n}(\hbar) \tag{5.3.13}$$

where, unlike (4.14), the characteristic Floquet index $\tilde{\Omega}_1$ is now normalized by the condition

$$\arg z(\tau_1 + T_1) = \arg z(\tau_1) + \tilde{\Omega}_1 T_1 - 2\pi. \tag{5.3.14}$$

So, let conditions (5.3.7) and (5.3.9) be fulfilled. Then, at each fixed $\hbar$ the family of invariant Lagrangian tori $\Lambda^2(\omega)$ with complex germ $r^3(\Lambda^2(\omega))$ is quantized, i.e. there appears a discrete set of geometric objects $[\Lambda^2(\omega_{n,l}), r^3(\Lambda^2(\omega_{n,l}))]$, where $\omega_{n,l} = (E^{(0)}_{n,l}(\hbar), \hbar l(\hbar))$. On each of them the canonical operator $K_{\Lambda^2(\omega_{n,l})}$ with complex phase is defined. Making use of them according to (5.2.12) a set of asymptotic eigenfunctions $\mathring{\Psi}_{E_{N,l}}(q, \hbar) = \mathring{\Psi}_{E_{n,l,\nu,\zeta}}(q, \hbar)$ corresponding to the series of eigenvalues

(5.3.11) is constructed. Thus, the quasi-classical spectral series of the Dirac operator $[\overset{\circ}{\Psi}_{E_{N,l}}(q,\hbar), E_{N,l}(\hbar)]$ corresponding to the family $[\Lambda^2(\omega_{n,l}), r^3(\Lambda^2(\omega_{n,l}))]$ is obtained.

From property (1) of subsection 5.2 it follows that each function $\overset{\circ}{\Psi}_{E_{n,l,\nu,\zeta}}(q,\hbar)$ is localized in the neighbourhood of the torus projection $\Lambda^2(\omega_{n,l})$ on $\mathbb{R}^3_q$. Also, we should make sure that, at a suitable choice of the normalization factor N_0, they satisfy (1.5) and, in this way, form a complete orthonormal set of states. Omitting unessential details, only the key features of the proof will be given. It is not difficult to check that the functions $\Xi_\nu(q_2,\tau_1)$ (4.16) obey the relation

$$\int \mathrm{d}q_2\, \overset{*}{\Xi}_{\nu'}(q_2,\tau_1)\Xi_\nu(q_2,\tau_1) = \delta_{\nu\nu'}(\pi\hbar)^{1/2} \tag{5.3.15}$$

and make up a complete set. The latter follows essentially from the completeness of the orthonormal set of the Hermite functions $U_\nu(\xi) = c_\nu \exp(-\xi^2/2)H_\nu(\xi)$. Using the stationary phase approximation (5.3.5) and (5.3.15) one obtains

$$\begin{aligned}\langle\overset{\circ}{\Psi}_{E_{N',l'}} \mid \overset{\circ}{\Psi}_{E_{N,l}}\rangle_{\mathrm{D}} = N_0^2[2\pi(\pi\hbar)^{1/2}\delta_{ll'}\delta_{\nu\nu'}\delta_{\zeta\zeta'} \\ \times \int_0^{T_1} \mathrm{d}\tau_1\{(e_1+e_3)^2+(e_2+e_4)^2+2(e_1+e_3)(e_2+e_4)\}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\Delta E\tau_1} + \mathrm{O}(\hbar)].\end{aligned} \tag{5.3.16}$$

Since

$$\sum_{j=1}^{4} e_j(\tau_1) = 1$$

the expression in the curly brackets in the integrand is identically equal to unity. Also, when the quantum numbers coincide $l = l'$, $\nu = \nu'$, $\zeta = \zeta'$ we have

$$\Delta E = \hbar\frac{\partial E^{(0)}_{n,l}(\hbar)}{\partial(\hbar n(\hbar))}(n-n') + \mathrm{O}(\hbar^2). \tag{5.3.17}$$

But from (5.3.9) follows $\partial E^{(0)}_{n,l}(\hbar)/\partial(\hbar n(\hbar)) = 2\pi/T_1$. The proof of this relationship is given in [7].

Thus, equation (5.3.16) takes the form

$$\langle\overset{\circ}{\Psi}_{E_{N',l'}} \mid \overset{\circ}{\Psi}_{E_{N,l}}\rangle_{\mathrm{D}} = N_0^2[2T_1\pi^{3/2}\hbar^{1/2}\delta_{NN'}\delta_{ll'} + \mathrm{O}(\hbar)] \tag{5.3.18}$$

from which at $N_0 = [2T_1\pi^{3/2}\hbar^{1/2}]^{-1/2}$ follows the validity of (1.5).

6. Quasi-classical spectral series of the Dirac operator in specific configuration fields

In this section we shall consider two examples of the three-degree-of-freedom relativistic Hamiltonian systems allowing families of two-dimensional invariant tori. The first one deals with a completely integrable case of the motion of an electron in the Coulomb field. Since the eigenvalues problem for the Dirac operator in the Coulomb field is solved exactly, with this example, by comparing quantum numbers, it is possible to separate that part of the exact spectrum to which an electron motion along the two-dimensional Lagrangian tori corresponds.

The second example refers to the case of a partially integrable system which describes the motion of a relativistic electron in an electromagnetic field with axial symmetry. The above system permits a family of two-dimensional invariant tori surrounding closed orbits of stable periodic motions.

6.1. Quantization of invariant two-dimensional Lagrangian tori in Coulomb field

In the Coulomb field with potential $A_0 = \alpha/\rho$, $\alpha = Ze_0$ the Hamiltonian is

$$\lambda^{(+)}(p,q) = -\frac{\alpha e_0}{\rho} + c(p_\rho^2 + \rho^{-2}p_\theta^2 + \rho^{-2}\sin^{-2}\theta p_\varphi^2 + m^2c^2)^{1/2} = -\frac{\alpha e_0}{\rho} + \varepsilon. \tag{6.1.1}$$

It is well known that in satisfying

$$A > 0 \qquad B > 0 \qquad C > 0 \qquad B^2 > AC \tag{6.1.2}$$

where

$$A = m^2c^2 - \frac{E_0^2}{c^2} \qquad B = \frac{\alpha e_0 E}{c^2} \qquad C = I_0^2 - \frac{\alpha^2 e_0^2}{c^2} \tag{6.1.3}$$

the Hamiltonian system (1.2) permits a family of invariant two-dimensional tori. They lie in the coordinate plane $\mathcal{T} = \{(p,q) : p_\theta = 0, \theta = \pi/2\}$ and have the form of (4.3)–(4.5) where

$$\Lambda^1(E_0, I_0) = \left\{(p_\rho, \rho) : -\frac{\alpha e_0}{\rho} + c(p_\rho^2 + \rho^{-2}I_0^2 + m^2c^2)^{1/2} = E_0\right\}. \tag{6.1.4}$$

The projection of $\Lambda^2(E_0, I_0)$ onto configuration space defining the classically allowed region lies in the equatorial plane and forms a ring bounded by two caustic circles

$$S_\pm^1(E_0, I_0) = \{(\rho, \theta, \varphi) : \rho = R_\pm(E_0, I_0), \theta = \pi/2, \varphi \in [0, 2\pi]\} \tag{6.1.5}$$

where $R_\pm$ are the classical turning points ($\dot\rho(R_\pm) = 0$):

$$R_\pm = \frac{B}{A} \pm \frac{\sqrt{B^2 - AC}}{A}. \tag{6.1.6}$$

Further, it is convenient to express the energy E_0 in terms of adiabatic invariants which, in our case, are

$$\begin{aligned} I_\varphi &= \frac{1}{2\pi}\int_0^{2\pi} p_\varphi \, d\varphi = I_0 \\ I_\rho &= \frac{1}{\pi}\int_{R_-}^{R_+} p_\rho \, d\rho = \frac{B}{\sqrt{A}} - \sqrt{C}. \end{aligned} \tag{6.1.7}$$

Then, we obtain

$$E_0(I_\rho, I_\varphi) = mc^2 \left[1 + \frac{\alpha^2 e_0^2}{c^2\left(I_\rho + \sqrt{I_\varphi^2 - \frac{\alpha^2 e_0^2}{c^2}}\right)^2}\right]^{-1/2}. \tag{6.1.8}$$

Performing the differentiations with respect to I_ρ and I_φ, one can find the radial and orbital frequencies, respectively:

$$\omega_\rho = \partial E_0/\partial I_\rho = A^{3/2}/(\alpha e_0 m^2) \tag{6.1.9}$$

$$\omega_\varphi = \partial E_0/\partial I_\varphi = I_0\omega_\rho/\sqrt{C}. \tag{6.1.10}$$

According to quantization conditions (5.3.7) and (5.3.9) we should take adiabatic invariants (6.1.7) to be given by

$$I_\varphi = \hbar l(\hbar) \qquad I_\rho = \hbar(n(\hbar) + \tfrac{1}{2}). \tag{6.1.11}$$

Whence, due to (6.1.8), we obtain the expression for the leading term of the energy spectrum

$$E^{(0)}_{n,l}(\hbar) = E_0[\hbar(n(\hbar)+\tfrac{1}{2}), \hbar l(\hbar)]$$
$$= mc^2 \left[1 + \frac{\alpha^2 e_0^2}{c^2 \left(\hbar(n(\hbar)+\frac{1}{2}) + \sqrt{\hbar^2 l^2(\hbar) - \frac{\alpha^2 e_0^2}{c^2}} \right)^2} \right]^{-1/2}. \tag{6.1.12}$$

We now calculate the energy correction $E^{(1)}_{n,l,\nu,\zeta}(\hbar)$ defined in (5.3.10). In our case the reduced system in variations (4.11) has the form

$$\left\{ \frac{\mathrm{d}}{\mathrm{d}\tau_1} - \frac{c^2}{\varepsilon\rho^2(\tau_1)} \begin{pmatrix} 0 & -I_0^2 \\ 1 & 0 \end{pmatrix} \right\} \begin{pmatrix} w(\tau_1) \\ z(\tau_1) \end{pmatrix} = 0. \tag{6.1.13}$$

Using (4.12) one can obtain the expression for the Floquet solution $z(\tau_1)$

$$z(\tau_1) = \frac{\mathrm{e}^{\mathrm{i}\theta}}{\sqrt{I_0}} \exp\left(\mathrm{i} I_0 \int_{\overset{\circ}{\tau}_1}^{\tau_1} \frac{c^2}{\varepsilon\rho^2(\tau_1)} \, \mathrm{d}\tau_1 \right) \qquad \theta \in \mathbb{R}. \tag{6.1.14}$$

For the characteristic Floquet index which can be defined from (4.13) and (4.14) we have

$$\Omega_1 = \frac{I_0}{T_1} \int_0^{T_1} \frac{c^2}{\varepsilon\rho^2(\tau_1)} \, \mathrm{d}\tau_1 = \omega_\varphi. \tag{6.1.15}$$

Here T_1 is the rotational period along the closed curve $\Lambda^1(E_0, I_0)$ equal to

$$T_1 = \frac{2}{c^2} \int_{R_-}^{R_+} \mathrm{d}\rho \frac{\varepsilon}{\sqrt{-A + \frac{2B}{\rho} - \frac{C}{\rho^2}}} = \frac{2\pi}{\omega_\rho}. \tag{6.1.16}$$

Define now the frequencies ω_ζ^s, $\zeta = \pm 1$, due to the interaction of the electron spin with the external field. In the Coulomb field polarization vector (5.1.30) is equal to $\boldsymbol{\mathcal{B}}(\tau_1) = (0, 0, \mathcal{B}(\tau_1))$, where $\mathcal{B}(\tau_1) = \alpha e_0 c^2 I_0 / [2\varepsilon^2(1+\gamma^{-1})\rho^3(\tau_1)]$. In this case (5.1.32) permits a set of two Floquet solutions

$$v_\zeta(\tau_1) = \exp\left\{ -\mathrm{i}\zeta \int_{\overset{\circ}{\tau}_1}^{\tau_1} \mathcal{B}(\tau_1) \, \mathrm{d}\tau_1 \right\} u_\zeta \qquad \zeta = \pm 1 \tag{6.1.17}$$

where

$$u_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad u_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

From this we obtain the expression for the characteristic Floquet index

$$\omega_{\zeta}^{\varepsilon} = \frac{\zeta}{T_1}\int_0^{T_1} \mathcal{B}(\tau_1)\,\mathrm{d}\tau_1 = \frac{\zeta}{2}\omega_\rho\left(-1+\frac{I_0}{\sqrt{C}}\right). \tag{6.1.18}$$

After substituting (6.1.15) and (6.1.18) into (5.3.10) we have $E^{(1)}_{n,l,\nu,\zeta}(\hbar) = \omega_\varphi\{\nu + \frac{1}{2}(1+\zeta)\} - (\zeta/2)\omega_\rho$. Hence, using (5.3.11) and (6.1.12) we find the following expression (with an accuracy to $\mathrm{O}(\hbar^2)$) for the quasi-classical energy spectrum

$$\begin{aligned}E_{n,l,\nu,\zeta}(\hbar) &= E_0[\hbar\{n(\hbar)+\tfrac{1}{2}(1-\zeta)\}, \hbar\{l(\hbar)+\nu+\tfrac{1}{2}(1+\zeta)\}] + \mathrm{O}(\hbar^2)\\ &= mc^2\left[1+\frac{\alpha^2 e_0^2}{c^2\left(\hbar\left\{n(\hbar)+\frac{1}{2}(1-\zeta)\right\}+\sqrt{\hbar^2\left\{l(\hbar)+\nu+\frac{1}{2}(1+\zeta)\right\}^2-\frac{e_0^2\alpha^2}{c^2}}\right)^2}\right]^{-1/2}\\ &\quad + \mathrm{O}(\hbar^2).\end{aligned} \tag{6.1.19}$$

The formula obtained allows us to establish a correspondence between the quantum numbers of the exact and quasi-classical energy spectra. Namely, let n_ρ and n_l denote, respectively, the radial and orbital quantum numbers entering the explicit formula for the hydrogen-like atom spectrum. Then, assuming the condition $n_\rho \sim 1/\hbar$, $n_l \sim 1/\hbar$ we have

$$n_\rho = n + \tfrac{1}{2}(1-\zeta), n_l = l + \nu. \tag{6.1.20}$$

Thus, as should have been expected, at $\hbar \to 0$ a highly excited spectral range both with respect to orbital n_l and radial n_ρ quantum numbers corresponds to the motion of a relativistic electron along the quantized two-dimensional invariant tori in the Coulomb field. However, it is interesting to note, that despite the fact that (6.1.19) was obtained on the assumption of great $n(\hbar) \sim 1/\hbar$, nevertheless, taking formally $\hbar n(\hbar) = \mathrm{O}(\hbar)$ (i.e. considering n to be small), we obtain the quasi-classical spectral series corresponding to the motion of an electron along an equilibrium circle [7].

6.2. Asymptotic series of eigenvalues for the Dirac operator in axially symmetric magnetic field with weak focusing corresponding to the motion of electron along two-dimensional Lagrangian tori surrounding equilibrium orbits

Consider the motion of a relativistic electron in the inhomogeneous axially symmetric magnetic field whose potentials in cylindrical coordinates (ρ, φ, z) take on the values

$$A_0 = A_\rho = A_z = 0 \qquad A_\varphi = \frac{b\rho^{2-q}}{2-q}\left[1+\frac{q(2-q)}{2}\frac{z^2}{\rho^2}\right] \tag{6.2.1}$$

where q is the focusing parameter, $0 < q < 1$, and $b =$ constant. The classical Hamiltonian is

$$\lambda^{(+)}(p,q) = c\left(p_\rho^2 + \rho^{-2}\left(p_\varphi - \frac{e_0}{c}A_\varphi\right)^2 + p_z^2 + m^2c^2\right)^{1/2} \tag{6.2.2}$$

and, as follows from Hamiltonian system (1.2), in the region of phase space $\mathcal{T} = \{(p, q) : p_z = z = 0\}$, determines a family of invariant two-dimensional Lagrangian manifolds described by a set of the canonical equations

$$\dot{\rho} = \frac{c^2}{E_0} p_\rho \qquad \dot{p}_\rho = \frac{P_\varphi}{\rho}\left(\frac{c^2 P_\varphi}{E_0 \rho^2} + \frac{e_0 c}{E_0} H(\rho)\right) \tag{6.2.3}$$

$$\dot{\varphi} = \frac{c^2 P_\varphi}{E_0 \rho^2} \qquad p_\varphi = I \tag{6.2.4}$$

with the initial data $\rho(0) = \rho_0$, $p_\rho(0) = p_{\rho 0}$, $\varphi(0) = \tau_2 (\mathrm{mod}\, 2\pi)$. Here

$$P_\varphi = I - \frac{e_0}{c}\frac{\rho^{2-q}}{2-q} \qquad \text{and } H(\rho) = \frac{b}{\rho^q}$$

is the magnitude of the magnetic field at the point ρ. Equations (6.2.3) and (6.2.4) can be integrated by quadratures if we make use of the energy integral $E_0^2/c^2 = m^2c^2 + p_\rho^2 + \rho^{-2} P_\varphi^2$. As a result, we obtain

$$\begin{aligned} p_\rho(\tau_1) &= \left(\frac{E_0^2}{c^2} - m^2c^2 - \frac{P_\varphi^2(\tau_1)}{\rho^2(\tau_1)}\right)^{1/2} \\ \varphi(\tau) &= \frac{c^2}{E_0}\int_0^{\tau_1} \frac{P_\varphi(t)}{\rho^2(t)}\,\mathrm{d}t + \tau_2 \end{aligned} \tag{6.2.5}$$

where the function $\rho(\tau_1)$ is implicity defined by the relationship

$$\tau_1 = \int_{\rho_0}^{\rho} \frac{c^2}{E_0}\frac{\mathrm{d}\rho}{p_\rho(\rho)}.$$

The period T_1 of the function $\rho(\tau_1)$ is given by

$$T_1 = 2\int_{R_-(E_0, I)}^{R_+(E_0, I)} \frac{E_0}{c^2}\frac{\mathrm{d}\rho}{p_\rho(\rho)}$$

where $R_\pm(E_0, I)$ are the roots of the equation $p_\rho(\rho) = 0$ at the given values of E_0 and I.

In a general case system in variations (4.11) corresponding to the family of Lagrangian tori (6.2.3), (6.2.4) failed to be integrated. We shall restrict ourselves to considering the Lagrangian tori surrounding equilibrium closed orbits of system (6.2.3), (6.2.4), and lying in a rather small neighbourhood of the latter. In this case all the calculations are carried out in an explicit form.

As follows from (6.2.3) and (6.2.4) the path of the electron in the plane xy is described by the equations

$$\ddot{\rho} + \frac{1}{\rho}(\dot{\rho}^2 - c^2\beta^2) = \frac{e_0 c}{E_0} H(\rho)\sqrt{c^2\beta^2 - \dot{\rho}^2} \tag{6.2.6}$$

$$\dot{\rho}^2 + \rho^2\dot{\varphi}^2 = c^2\beta^2 \tag{6.2.7}$$

where $c\beta$ is the velocity of an electron motion. Introduce the following notation

$$\omega_0 = -\frac{e_0 c}{E_0} H(R) \qquad \varepsilon = \frac{1}{R}\sqrt{(\rho_0 - R)^2 + \frac{\dot{\rho}_0^2}{\omega_\Delta^2}} \tag{6.2.8}$$

where

$$\omega_\Delta = \sqrt{1-q}\,\omega_0,\ \dot{\rho}_0 = \dot{\rho}(0) = \frac{c^2}{E_0} p_{\rho 0}.$$

Here, ω_0 stands for the revolution frequency of an electron along the equilibrium circle l_R, whose radius R will be defined below. The constant ε characterizes the amount of small deviations of the path (6.2.6), (6.2.7) away from l_R, and below it will be considered to be a dimensionless expansion parameter.

The solution of (6.2.6) will be searched for as a power series in ε

$$\rho(\tau_1) = R + \varepsilon\rho_1(\tau_1) + \varepsilon^2\rho_2(\tau_1) + \mathrm{O}(\varepsilon^3). \tag{6.2.9}$$

Inserting (6.2.9) into (6.2.6) we obtain, with an accuracy to order $\mathrm{O}(\varepsilon^3)$, the following results

$$c\beta = \omega_0 R \tag{6.2.10}$$

$$\rho(\tau_1) = R\left\{1 + \varepsilon\cos\theta + \varepsilon^2\left[-\frac{3+q}{6}\cos^2\theta + \frac{2q+3}{6} + A\sin\theta + B\cos\theta\right]\right\}. \tag{6.2.11}$$

Here, $\theta = \omega_\Delta\tau_1 - \alpha$, where the angle α is assumed to be equal to

$$\alpha = \cos^{-1}\left((\rho_0 - R)\Big/\sqrt{(\rho_0 - R)^2 + \frac{\dot{\rho}_0^2}{\omega_\Delta^2}}\right) \tag{6.2.12}$$

and A and B are the integration constants, which are defined by choosing initial conditions and, in our case, are

$$\begin{aligned} A &= -\sin\alpha\left(\frac{3+q}{6}\cos^2\alpha + \frac{2q+3}{6}\right) \\ B &= -\cos\alpha\left(\frac{3+q}{6}\cos^2\alpha - \frac{1}{2}\right). \end{aligned} \tag{6.2.13}$$

To define the value of the classical energy E_0 we shall use (6.2.7), (6.2.8) and (6.2.10). Whence, we obtain

$$E_0 = \sqrt{m^2c^4 + e_0^2 H^2(R) R^2}. \tag{6.2.14}$$

In turn, from (6.2.7), (6.2.10) and (6.2.11), follows the expression for the function $\dot{\varphi}(\tau_1)$

$$\dot{\varphi}(\tau_1) = \omega_0\left\{1 - \varepsilon\cos\theta + \varepsilon^2\left[\left(2 - \frac{q}{3}\right)\cos^2\theta - \left(1 - \frac{q}{6}\right) - A\sin\theta - B\cos\theta\right]\right\} + \mathrm{O}(\varepsilon^3). \tag{6.2.15}$$

In the same way, allowing the angular momentum of an electron in the first equation of (6.2.4) to be equal to $I = I_0 + I_1$ where $I_1 = \mathrm{O}(\varepsilon^2)$ and making use of (6.2.10) one obtains

$$R(I_0) = \left(-\frac{e_0 b}{cI_0}\frac{1-q}{2-q}\right)^{1/(q-2)} \tag{6.2.16}$$

$$\frac{c^2 I_1}{E_0 R^2} = -\omega_0 \frac{1-q}{2}\varepsilon^2. \tag{6.2.17}$$

Now, consider the expression for the leading term of the energy spectrum $E^{(0)}_{n,l}(\hbar)$. *Inserting (6.2.11) into (5.3.9) we obtain, with an accuracy to* $\mathrm{O}(\varepsilon^3)$, *the following condition*

$$\frac{1}{2}\frac{E_0}{c^2}\sqrt{1-q}\,\omega_0\varepsilon^2 R^2 = \hbar(n+\tfrac{1}{2}). \tag{6.2.18}$$

By comparing (6.2.17) and (6.2.18) we find

$$I_1 = -\hbar\sqrt{1-q}(n+\tfrac{1}{2}). \tag{6.2.19}$$

From (5.3.7) and (6.2.19) it follows that I_0 takes on the quantized values

$$I_0(n,l) = \hbar l + \hbar\sqrt{1-q}(n+\tfrac{1}{2}). \tag{6.2.20}$$

On the other hand, due to (6.2.16) and (6.2.20) we obtain the quantization rule of the radius $R(n,l)$ of the equilibrium circle l_R. As a result, we have

$$E^{(0)}_{n,l}(\hbar) = \sqrt{m^2c^4 + e_0^2 H^2(R(n,l))R^2(n,l)} \tag{6.2.21}$$

where the integer parameters $l = l(\hbar)$ and $n = n(\hbar)$ obey the conditions

$$\lim_{\hbar\to 0}\hbar l(\hbar) = I \qquad \lim_{\hbar\to 0} E^{(0)}_{n,l}(\hbar) = E_0.$$

In taking the second step of our considerations we wish to calculate the value of the Floquet index Ω_1 (see (4.13), (4.14)). For this purpose, we consider (4.11) for the vector

$$\chi(\tau_1) = \begin{pmatrix} w(\tau_1) \\ z(\tau_1) \end{pmatrix}.$$

In our case

$$\dot\chi(\tau_1) = G(\tau_1)\chi(\tau_1) \qquad G(\tau_1) = \begin{pmatrix} 0 & \frac{e_0 q}{c}\dot\varphi(\tau_1)H(\rho(\tau_1)) \\ c^2/E_0 & 0 \end{pmatrix}. \tag{6.2.22}$$

We can now expand the matrix $G(\tau_1)$ in powers of ε and restrict ourselves to the terms of order ε^2. To construct the Floquet solution which satisfies (6.2.22) and (4.12) with an accuracy to order $\mathrm{O}(\varepsilon^3)$ we apply the perturbation theory. Finally, after simple but cumbersome calculations we arrive at the following result

$$\Omega_1 = \sqrt{q}\,\omega_0\left[1 + \frac{q(1-q)(2-q)}{4(1-5q)}\varepsilon^2\right] + \mathrm{O}(\varepsilon^3). \tag{6.2.23}$$

In order to determine the values of ω^s_ζ one needs to turn to (5.1.32). In the axially symmetric magnetic field with potentials (6.2.1), polarization vector (5.1.30) is equal to

$$\mathcal{B}(\tau_1) = \left(0, 0, -\frac{e_0 c}{2E_0} H(\rho(\tau_1))\right)$$

and (5.1.32) is the linear Hamiltonian system with T_1-periodic coefficients. The Floquet solutions of this system possess the following characteristic indices:

$$\omega^s_\zeta = -\frac{\zeta}{T_1}\int_0^{T_1} \frac{e_0 c}{2E_0} H(\rho(\tau_1))\,\mathrm{d}\tau_1. \tag{6.2.24}$$

Whence, with an accuracy to $\mathrm{O}(\varepsilon^3)$, we have

$$\omega^s_\zeta = \frac{\zeta}{2}\omega_0 + \mathrm{O}(\varepsilon^3) \qquad \zeta = \pm 1. \tag{6.2.25}$$

By summing up the results obtained above and allowing for the equality $\omega_0 = \partial E_0/\partial I_0$ (which follows from (6.2.14) and (6.2.16)) we obtain the expression for the quasi-classical energy spectrum

$$\begin{aligned} E_{n,l+\zeta/2,\nu}(\hbar) = \bigg[& m^2c^4 + e_0^2 H^2(R(n,l+\zeta/2))R^2(n,l+\zeta/2) - 2e_0 cH(R(n,l+\zeta/2)) \\ & \times \hbar\sqrt{q}(\nu+\tfrac{1}{2}) + \frac{c^2}{R^2(n,l+\zeta/2)}\frac{q\sqrt{q(1-q)}(2-q)}{1-5q}\hbar^2(n+\tfrac{1}{2})(\nu+\tfrac{1}{2})\bigg]^{1/2} \\ & + \mathrm{O}(\varepsilon^3) + \mathrm{O}(\hbar^2) \end{aligned} \tag{6.2.26}$$

where $R(n, l+\zeta/2) \equiv R(I_0(n, l+\zeta/2))$. As in the previous example, putting in (6.2.26) formally $\hbar n(\hbar) = \mathrm{O}(\hbar)$ we obtain the series of energy levels of an electron moving along an equilibrium orbit [7].

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Appendix 1

To prove (5.2.5) we make use of the stationary phase method. Consider a rapidly oscillating integral of the form

$$I(\hbar^{-1}) = \int \varphi(\omega, x)\exp\left[\frac{\mathrm{i}}{\hbar}S(\omega, x)\right]\mathrm{d}x \qquad x \in \mathbb{R}^l \tag{A1.1}$$

where ω are parameters. Let the function $S(\omega, x)$ possess only one non-degenerate critical point $x_0 = x_0(\omega)$, i.e. $S_x(x_0) = 0$, $S_{xx}(x_0) \neq 0$. Then, there occurs the following expansion of the integral (A1.1) at $\hbar \to 0$

$$I(\hbar^{-1}) = \sqrt{\frac{2\pi\hbar}{|S_{xx}(x_0)|}}\varphi(\omega, x_0)\exp\left[\frac{\mathrm{i}}{\hbar}S(\omega, x_0) + \frac{\mathrm{i}\pi}{4}\operatorname{sgn} S_{xx}(x_0)\right] + \mathrm{O}(\hbar). \tag{A1.2}$$

Now, consider the integrand in the right-hand side of (5.2.3) and separate out the rapidly oscillating part of the exponent in it. Taking into account the condition $\Delta q_2 = q_2 - q_2(\tau) \equiv q_2 = \mathrm{O}(\hbar^{1/2})$ and the identity $p_1(\tau_1^j(p_1)) \equiv p_1$ rewrite (5.2.3) as

$$(K(\Omega_{j_1})[\mathcal{F}])(q) = \frac{\exp\left(\frac{\mathrm{i}\pi}{2}\gamma_{j_1}\right)}{\sqrt{-2\pi\mathrm{i}\hbar}}\int \mathrm{d}p_1 \exp\left[\frac{\mathrm{i}}{\hbar}S(p_1, q)\right]\frac{f(\tau_1^{j_1}(p_1), q)}{\sqrt{|\dot{p}_1(\tau_1^{j_1}(p_1))|}} \tag{A1.3}$$

where

$$S(p_1, q) = p_1 q_1 + \int_{\overset{\circ}{\tau}_1}^{\tau_1^{j_1}(p_1)} \mathrm{d}t\, p_1(t)\dot{q}_1(t) - p_1 q_1(\tau_1^{j_1}(p_1)) + I_0\varphi.$$

Thus, the coordinates q_1 and φ in (A1.3) stand for the parameters. The stationary phase point is found from the condition $S_{p_1}(p_1, q) = 0$. Whence, we obtain $q_1 = q_1(\tau_1^{j_1}(p_1))$. In the overlap region $V_{j_1} \cap V_{j_2}$, this equation has a unique solution $p_1 = p_1(q_1)$, which does define the desired stationary point. It should be noted that the equality $\tau_1^{j_1}(p_1)|_{p_1=p_1(q_1)} = \tau_1^{j_2}(q_1)$ is valid for the functions $\tau_1^{j_1}(p_1)$ and $\tau_1^{j_2}(q_1)$ in the region $V_{j_1} \cap V_{j_2}$. Allowing for this for the stationary point we have

$$S(p_1(q_1), q) = \int_{\overset{\circ}{\tau}_1}^{\tau_1^{j_2}(q_1)} p_1(t)\,\mathrm{d}q_1(t) + I_0\varphi \tag{A1.4}$$

$$S_{p_1p_1}(p_1(q_1), q) = -\frac{\dot{q}_1}{\dot{p}_1}(\tau_1^{j_2}(q_1)). \tag{A1.5}$$

Applying formula (A1.2) to integral (A1.3) and allowing for (A1.4) and (A1.5) we obtain

$$\begin{aligned}(K(\Omega_{j_1})[\mathcal{F}])(q) &= \frac{\exp\left(\frac{\mathrm{i}\pi}{2}\gamma_{j_1}\right)}{\sqrt{|\dot{q}_1(\tau_1^{j_2}(q_1))|}}\exp\left\{\frac{\mathrm{i}}{\hbar}\left(\int_{\overset{\circ}{\tau}_1}^{\tau_1^{j_2}(q_1)} p_1(t)\,\mathrm{d}q_1(t) + I_0\varphi\right)\right\}\\ &\quad\times\exp\left\{\frac{\mathrm{i}\pi}{4}\left[1 - \operatorname{sgn}\frac{\dot{q}_1}{\dot{p}_1}(\tau_1^{j_2}(q_1))\right]\right\} f(\tau_1^{j_2}(q_1), q) + \mathrm{O}(\hbar)\\ &= \exp\left\{\frac{\mathrm{i}\pi}{2}\left(\gamma_{j_1} - \gamma_{j_2} + \operatorname{inerdex}\frac{\dot{q}_1}{\dot{p}_1}\right)\right\}(K(\Omega_{j_2})[\mathcal{F}])(q) + \mathrm{O}(\hbar)\end{aligned} \tag{A1.6}$$

where by definition

$$\operatorname{inerdex}\frac{\dot{q}_1}{\dot{p}_1} = -\frac{1}{2}\left(\operatorname{sgn}\frac{\dot{q}_1}{\dot{p}_1} - 1\right).$$

By direct checking it is not difficult to see the validity of the following equality:

$$\gamma_{j_2}[l(\overset{\circ}{\tau}_1, \tau_1)] = \gamma_{j_1}[l(\overset{\circ}{\tau}_1, \tau_1)] + \operatorname{inerdex}\frac{\dot{q}_1}{\dot{p}_1}(\tau_1) \qquad \tau_1 \in V_{j_1} \cap V_{j_2}. \tag{A1.7}$$

Equation (5.2.5) immediately follows from the latter and (A1.6).

Appendix 2

We are interested in the asymptotic behaviour of the wavefunctions $\overset{\circ}{\Psi}_E(q,\hbar)$ at $\hbar \to 0$ in a small domain of the caustic $\pi_q(\Sigma) \subset \mathbb{R}^3_q$ formed, in our case, from the points $\{q_1(\tau_1^\pm) = R_\pm, q_2 = 0, \varphi \in [0, 2\pi]\}$, where $\dot{q}_1(\tau_1^\pm) = 0$. Choose the partition of unity so that

$$\begin{aligned} \operatorname{supp} e_2(\tau_1, \varepsilon) &= [\tau_1^+ - \varepsilon, \tau_1^+ + \varepsilon] \\ \operatorname{supp} e_4(\tau_1, \varepsilon) &= [\tau_1^- - \varepsilon, \tau_1^- + \varepsilon] \end{aligned} \tag{A2.1}$$

where ε is a small parameter of order $O(\hbar^{1/6})$. For the sake of simplicity we assume $e_\zeta = (e_2, e_4)$, $\tau_1^\zeta = (\tau_1^+, \tau_1^-)$. Then, the function $\overset{\circ}{\Psi}_E(q,\hbar)$ at $\hbar \to 0$ takes the form (equation (5.3.1) is assumed to be fulfilled)

$$\begin{aligned} \overset{\circ}{\Psi}_E(q,\hbar) &= \int_0^{T_1} \mathrm{d}\tau_1 (e_1(\tau_1,\varepsilon) + e_3(\tau_1,\varepsilon))\sqrt{\dot{q}_1(\tau_1)}\left(\exp\frac{\mathrm{i}}{\hbar}\int_0^{\tau_1} p_1(t)\,\mathrm{d}q_1(t)\right) \\ &\quad \times \delta(q_1 - q_1(\tau_1))\Phi_\nu(\tau_1, q_2, \varphi) + \sum_\zeta \int_{\tau_1^\zeta - \varepsilon}^{\tau_1^\zeta + \varepsilon} \mathrm{d}\tau_1 \frac{e_\zeta(\tau_1,\varepsilon)}{\sqrt{-2\pi\mathrm{i}\hbar}}\sqrt{\dot{p}_1(\tau_1)} \\ &\quad \times \left\{\exp\frac{\mathrm{i}}{\hbar}\left[\int_0^{\tau_1} p_1(t)\,\mathrm{d}q_1(t) + p_1(\tau_1)(q_1 - q_1(\tau_1))\right]\right\}\Phi_\nu(\tau_1, q_2, \varphi) \\ &\equiv J_0 + \sum_\zeta J_\zeta. \end{aligned} \tag{A2.2}$$

At q_1 near to $q_1(\tau_1^\zeta)$ the main contribution into the asymptotic behaviour of functions (A2.2) is made by the two last integrals J_ζ. Let us analyse their asymptotic at $\hbar \to 0$. By changing to the new variable $u = (\tau_1 - \tau_1^\zeta)/\varepsilon^2$ one obtains

$$\begin{aligned} J_\zeta &= \varepsilon^2 \int_{-1/\varepsilon}^{1/\varepsilon} \mathrm{d}u \frac{e_\zeta(\tau_1^\zeta + \varepsilon^2 u)}{\sqrt{-2\pi\mathrm{i}\hbar}}\sqrt{\dot{p}_1(\tau_1^\zeta + \varepsilon^2 u)}\Phi_\nu(\tau_1^\zeta + \varepsilon^2 u, q_2, \varphi) \\ &\quad \times \exp\frac{\mathrm{i}}{\hbar}\left[\int_0^{\tau_1^\zeta + \varepsilon^2 u} p_1(t)\,\mathrm{d}q_1(t) + p_1(\tau_1^\zeta + \varepsilon^2 u)(q_1 - q_1(\tau_1^\zeta + \varepsilon^2 u))\right]. \end{aligned} \tag{A2.3}$$

Consider the region of configuration space containing the caustic $\pi_q(\Sigma)$ and consisting of the points

$$V_\delta^\zeta[\pi_q(\Sigma)] = \{(q) : q_1 - q_1(\tau_1^\zeta) = O(\hbar^{2/3}), q_2 = O(\hbar^{1/2}), \varphi \in [0, 2\pi]\}. \tag{A2.4}$$

Expand expression (A2.3) in powers of the parameter $\varepsilon^2 = O(\hbar^{1/3})$. Then, at points (A2.4) we have

$$\begin{aligned} J_\zeta &= \varepsilon^2 e_\zeta(\tau_1^\zeta)\sqrt{\dot{p}_1(\tau_1^\zeta)}\Phi_\nu(\tau_1^\zeta, q_2, \varphi) \\ &\quad \times \exp\frac{\mathrm{i}}{\hbar}\left[\int_0^{\tau_1^\zeta} p_1(t)\,\mathrm{d}q_1(t) + p_1(\tau_1^\zeta)(q_1 - q_1(\tau_1^\zeta))\right]\frac{1}{\sqrt{-2\pi\mathrm{i}\hbar}} \\ &\quad \times \left\{\int_{-\infty}^{\infty} \mathrm{d}u \exp\mathrm{i}\left[u\dot{p}_1(\tau_1^\zeta)\frac{q_1 - q_1(\tau_1^\zeta)}{\hbar^{2/3}} - \frac{u^3}{6}\dot{p}_1(\tau_1^\zeta)\ddot{q}_1(\tau_1^\zeta)\right] + O(\hbar^{1/3})\right\}. \end{aligned} \tag{A2.5}$$

By making use of the integral representation of the Airy function

$$Ai(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\xi \exp \mathrm{i}(\xi x + \xi^3/3)$$

and the equality $e_\zeta(\tau_1^\zeta) = 1$ one obtains:

$$J_\zeta = \varepsilon^2 \hbar^{-1/2} \frac{\sqrt{2\pi}\exp(\mathrm{i}\pi/4)}{\left(-\frac{1}{2}\dot{p}_1(\tau_1^\zeta)\ddot{q}_1(\tau_1^\zeta)\right)^{1/3}} \sqrt{\dot{p}_1(\tau_1^\zeta)}\Phi_\nu(\tau_1^\zeta, q_2, \varphi)$$
$$\times \left(\exp \frac{\mathrm{i}}{\hbar}\left[\int_0^{\tau_1^\zeta} p_1(t)\,\mathrm{d}q_1(t) + p_1(\tau_1^\zeta)(q_1 - q_1(\tau_1^\zeta))\right]\right)$$
$$\times \left\{ Ai\left[Z_\zeta \frac{q_1 - q_1(\tau_1^\zeta)}{\hbar^{2/3}}\right] + \mathrm{O}(\hbar^{1/3})\right\} \qquad \text{(A2.6)}$$

where $Z_\zeta = \dot{p}_1(\tau_1^\zeta)/(-\frac{1}{2}\dot{p}_1(\tau_1^\zeta)\ddot{q}_1(\tau_1^\zeta))^{1/3}$. Thus, in the neighbourhood of the caustic $V_\delta^\zeta[\pi_q(\Sigma)]$ the leading term of asymptotic (A2.6) has the order $\mathrm{O}(\hbar^{-1/6})$, which implies a significant amplitude growth of the solution $\overset{\circ}{\Psi}_E(q, \hbar)$ at these points at $\hbar \to 0$.

Consider now the region of the shadow of configuration space formed by the points

$$W_\delta^\zeta = \{(q) : q_1 - q_1(\tau_1^\zeta) = \mathrm{O}(\hbar^\alpha), q_2 = \mathrm{O}(\hbar^{1/2}), \varphi \in [0, 2\pi]\} \qquad \text{(A2.7)}$$

where $\alpha < \frac{1}{2}$ and $q_1 - q_1(\tau_1^\zeta) \neq 0$. We shall show that in the region W_δ^ζ the function $\overset{\circ}{\Psi}_E(q, \hbar)$ is an asymptotic zero. As in the previous case the behaviour of the function $\overset{\circ}{\Psi}_E(q, \hbar)$ at points (A2.7) is governed by the integrals J_ζ. Make the change of variables $u = (\tau_1 - \tau_1^\zeta)/\varepsilon^3$ and rewrite in the form

$$J_\zeta = \mu(\hbar)\int_{-1/\varepsilon^2}^{1/\varepsilon^2} \mathrm{d}u \left[\exp \frac{\mathrm{i}}{\hbar}\varepsilon^3 u \dot{p}_1(\tau_1^\zeta)(q_1 - q_1(\tau_1^\zeta))\right] \tilde{\Phi}_\nu(u, q_2, \varphi, \hbar) \quad \text{(A2.8)}$$

where

$$\mu(\hbar) = \frac{\varepsilon^3}{\sqrt{-2\pi \mathrm{i}\hbar}} \exp \frac{\mathrm{i}}{\hbar}\left[\int_0^{\tau_1^\zeta} p_1(t)\,\mathrm{d}q_1(t) + p_1(\tau_1^\zeta)(q_1 - q_1(\tau_1^\zeta))\right] = \mathrm{O}(1) \qquad \text{(A2.9)}$$

and

$$\tilde{\Phi}_\nu(u, q_2, \varphi, \hbar) = e_\zeta(\tau_1^\zeta + \varepsilon^3 u)\sqrt{\dot{p}_1(\tau_1^\zeta + \varepsilon^3 u)}\Phi_\nu(\tau_1^\zeta + \varepsilon^3 u, q_2, \varphi)$$
$$\times \exp \frac{\mathrm{i}}{\hbar}\left\{ \int_{\tau_1^\zeta}^{\tau_1^\zeta + \varepsilon^3 u} p_1(t)\,\mathrm{d}q_1(t) + [p_1(\tau_1^\zeta + \varepsilon^3 u) - p_1(\tau_1^\zeta) - \dot{p}_1(\tau_1^\zeta)\varepsilon^3 u]\right.$$
$$\left. \times (q_1 - q_1(\tau_1^\zeta)) + p_1(\tau_1^\zeta + \varepsilon^3 u)[q_1(\tau_1^\zeta) - q_1(\tau_1^\zeta + \varepsilon^3 u)]\right\}. \qquad \text{(A2.10)}$$

Expanding function (A2.10) in an asymptotic series in ε^3, we find that the leading term of the expansion is of the order O(1). In view of the condition $\tilde{\Phi}_\nu|_{u=\pm 1/\varepsilon^2} = 0$ (which immediately follows from (A2.1)) and the function $\tilde{\Phi}_\nu$ being finite, the integration limits in (A2.8) may be replaced by $\pm\infty$. Since in the region (A2.7) integrals (A2.8) do not have critical points, and function (A2.10) is regular with respect to the variable u, then, according to the theory of asymptotic estimations of rapidly oscillating integrals one obtains

$$J_\zeta = \mathrm{O}(\hbar^\infty). \qquad \text{(A2.11)}$$

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